INPUT-OUTPUT ANALYSIS AND ITS APPLICATIONS

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PREFACE

Input–Output analysis is a technique designed by mathematicians and usually explained in highly sophisticated mathematical terminology. For this reason the number of users tends to be limited to those who have a fairly advanced mathematical background. Unfortunately many economists who could profitably use input–output methods are non-mathematicians and hence are unable to read the literature on the subject or use this powerful analytical tool. This is a great pity because a high degree of mathematical knowledge is not necessary for an understanding of how the system works, and indeed if an economist were familiar with the system he could use it efficiently in the solution of many important practical problems without having recourse to mathematics at all. In this connexion it might be mentioned that the transactions table alone is a powerful tool for the budgeting of alternative systems within an economy, so as to determine whether policy recommendations are feasible or otherwise.*

Once one proceeds beyond the transactions-table stage, however, some knowledge of matrix algebra is required but even here a small amount of mathematics can go a long way, indeed if the basic algebraic requirements are mastered, the services of a mathematician can be enlisted to help out with the more difficult points that arise. From experience in teaching the subject the authors have found that for non-mathematicians in particular the initial stages of learning the input–output techniques are very off-putting and most of the recognized text books are of little help to these people. The mathematical notation used seems to cause a mental block so that many students are beaten even before they start. For these students an elementary text book is needed which spells out the various steps in some detail and in which symbols are kept to a minimum. The present work has been written for such students and for practising economists who may wish to learn something of the system without having a teacher. The treatment here is most elementary but, having used an earlier draft with University-level students of economics, the authors are satisfied that it fulfills its purpose adequately. Once the students have mastered the basic principles outlined, most of them have little difficulty in going on to study the more orthodox text books.

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* For an example of this application see:
O'Connor, R. "An analysis of recent policies for beef and milk." *Journal of the Statistical and Social Inquiry Society of Ireland,* Vol. XXIII, Pt. II, 1969–70 issue.

The work is divided into two parts, a main part and a mathematical appendix. The main part is further subdivided into three sections. The first of these discusses the input–output concept, and with the aid of examples from a number of countries explains the construction of the different types of tables. The method of using these tables in planning an economy and in calculating economic multipliers is also discussed in this section.

The second section contains a review of input–output methodology showing in particular the method of pricing commodities, the treatment of imports and exports and a discussion of the methods used in dealing with joint products. This section also explains the use of artificial rows and columns and the updating of coefficients via Stone's RAS method or otherwise.

The third section deals with applications of the input–output system in the solution of practical problems. The first application shows how partial transactions tables were used to examine Irish milk and beef policies in order to see if a shift from milk to beef production would ease somewhat the pre-E.E.C. milk subsidy burden. It is suggested that this technique could be used widely to examine a variety of problems in both the agricultural and industrial field. The second example shows how input–output can be used to examine an import substitution problem taking as an example Irish grain support-price policies. The final chapter in this section shows how the method of linear programming may be applied to input–output models. Three numerical examples of the application of this technique are given, each example having a highly aggregated British, Netherlands or Irish input–output table as a basis for a linear programming experiment.

Most of the main part of the book can be understood without any knowledge of advanced mathematics. The small amount of matrix algebra used is explained in the text. One cannot get very far with input–output analysis, however, without some knowledge of matrix algebra and for this reason we have included a mathematical appendix dealing with this subject. The treatment here is not exhaustive but it does cover most of the mathematics needed for a more detailed study of the ideas and concepts presented in the earlier chapters of the book. Furthermore it is written in such a way that it can be understood by non-mathematicians. For those who would wish to go into the subject of matrix algebra in greater depth (and at not too high a level) we would recommend a recent book by M.E. Yaari which is fully referenced at the end of the mathematical appendix.

It remains for us to acknowledge the help we have received from colleagues in the Economic and Social Research Institute who read earlier drafts of this work and made most useful suggestions for improvement. Our special thanks are due to Miss Brenda Forde for her patience and skill in typing the manuscript and for correcting numerous errors.

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Section 3 APPLICATIONS OF THE INPUT–OUTPUT SYSTEM

INTRODUCTION

Input–output analysis is concerned with studying the interdependence of the producing and consuming units in a modern economy and with showing the interrelations among different sectors which purchase goods and services from other sectors and which in turn produce goods and services which are sold to other sectors. In order to make such a study, the various economic flows are set out in an input–output table which is specially designed to provide a concise and systematic arrangement of all economic activity within a state or region.

If all sectors are considered as being both producers and consumers the system is represented by what is called a *closed model.* In such a model, households constitute an industry whose output is labour and whose inputs are consumption goods. Models of this kind are usually traced back to the Tableau Economique of Quesnay published in 1758[1] but the inspiration of modern work in this field owes much to the general equilibrium system of Leon Walras.[2] Closed models do not lend themselves readily to algebraic manipulation since they are completely circular with no exogenous variables. Nevertheless they have great analytic merit as has been demonstrated by a number of writers in recent years. [3] [4]

It was always recognized that the object of economic activity was satisfaction of final demand, and to Leontief[5] [6] goes the credit for exploiting this fact in his design of what is called the "open static model". In this system final demand, i.e. exports, Government services, household consumption and capital formation, is assumed to be related to other sectors but is autonomously determined by factors outside the system. Labour is considered as an input but not as a functionally related product of households. Open (Leontief) models are the ones now in common use everywhere.

Leontief commenced his research on an empirical model of the United States economy in 1931 and published his first results in 1936. One of the most valuable results of this study was to stimulate empirical work on input–output relations in a number of countries, though in the early years the major emphasis was on assembling the accounts rather than on the analytic aspects of the model. This emphasis however must not be underestimated, since an input–output table may be viewed independently of any functional relations, as a comprehensive, detailed and consistent framework for organizing economic statistics.[7] In the process of assembling the table, inconsistencies, gaps and redundancies in

the statistical system of an economy are revealed and the production of input–output tables has no doubt led to improved national statistics in the countries where such tables are prepared.

The first input–output model for the Irish economy was made in the Central Statistics Office, for 1956, and consisted of 36 productive sectors. A similar 36-sector model was constructed for 1960, and a 92-sector model for 1964 was published in 1970[8], while a 33-sector model for 1968 was published in 1972.[9] An aggregated 9-sector version of the 1960 table was published by Geary in 1964[10], and a modified version of this table is shown in Table 1.1.

The accumulation of statistical material in the input–output framework has suggested alternative techniques of analysis and there is now a considerable variety of input–output models in use. Among these the more important are dynamic and mathematical programming models which offer exciting possibilities in economic research.[11],[12] and [13]

The methodology of input–output is still undergoing a process of revision and change, as is to be expected for a new and developing branch of economic science. The reader who intends to study the subject beyond the scope of this textbook should examine two fairly recent publications[14],[15], which reveal some differences in methods of pricing and valuation, as compared with those described in Chapter 4 of this book. It is unlikely that anything like a final stage has yet been reached, in methodology. The reader would be well advised to keep this in mind and to expect possible changes in methodology, over intervals of five or ten years. The last word on this aspect of input–output analysis has certainly not yet been written.

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Section 1

The Input—Output System

1 EXPLAINING THE INPUT–OUTPUT SYSTEM

In input–output analysis the computational procedures are based on matrix algebra but a knowledge of this subject is not necessary for an elementary understanding of how the system works. Lack of advanced mathematical knowledge is not the main difficulty in this field of enquiry. Assembling the basic data from various sources presents the greatest problem and occupies most of the time involved in the operation. The final calculations, involving matrix inversion, can be made in a very short time and at relatively small cost on a computer.

Though the workings of the system can be understood without a knowledge of matrix algebra some familiarity with this subject is necessary for those who wish to do any worthwhile analysis based on input–output. For this reason a short explanation of the relevant elements of matrix algebra is given in the Mathematical Appendix at the end of the book and students who wish to work with the system should study this appendix before reading the main text.

INPUT–OUTPUT TABLES

In doing an input–output study it is necessary to produce three main tables,
1. A transactions table
2. A table of technical coefficients, and
3. A table of inter-dependence coefficients sometimes called total coefficients.

Each of these tables is described below.

Transactions Table

The basic table of the input–output system is known as the *Transactions Table* in which are entered in value terms the various economic flows within the economy during some particular base year. In order to prepare the table the economy is divided up into a number of sectors based usually on Census of Production and other National statistical classifications. Output of each sector is distributed along a row of the table while the corresponding column records the inputs of this sector, see Table 1.1.

Transactions tables have a number of special features but before going on to discuss these we will examine a row and column of Table 1.1 to show in a general way how the entries are derived.

The first row of this table shows how the output of the agricultural sector was disposed of in 1960. The entry (in £ million) of 2·180 in the first column represents mainly agricultural seeds sold off farms and later repurchased by farmers. The entry of 77·000 in Column (2) is the value of animals sold to meat factories for slaughter, the value of home grown grain used in milling and of vegetables and other crops used for processing. The entry of 3·104 in Column (3) represents mainly the value of malting barley sold to brewers, malsters and distillers. The entry of 0·550 in Column (4) is the value of home produced wool used in the Irish textile and apparel industry. The entry of 1·033 in Column (6) is mainly the value of timber used in the furniture industry, casualty hides used in the leather industry, wood used in the paper industry and potatoes and grain used in the chemical industry. The entry of 1·143 in Column (9) is the value of home-grown feed, mainly hay and grain, used by non-agricultural horses. The entry of 62·111 in Column (11) is the amount of agricultural produce consumed in households without undergoing any industrial processing. Included here is the value of liquid milk, farmers' butter, and eggs and vegetables consumed by persons in the State. Bread, flour, fresh meat, bacon and other such processed commodities consumed within the State are not included here. These are the products of the food processing sector and are shown in the Household Consumption column of this sector, i.e. in Column (11) of Row (2). The entry 0·803 in Column (12) is Government current expenditure on state forestry.

The entry of 2·671 in Column (13) of Row (1) represents mainly the value of changes in livestock on farms between the beginning and end of the year. It also contains changes in stocks of cereals held by Bord Grain, merchants and millers. Finally, the figure of 49·750 represents the value of unprocessed agricultural produce exported. The main items included here are live animals, raw wool, potatoes, unmilled cereals, fresh vegetables and fish. Processed agricultural produce exported is not included here but in other sectors such as the food processing and textile sectors. The total of the Agricultural etc. row in Column (16), which is 200·345, represents the official output of agriculture, forestry and fishing in that year. It should be mentioned in this connexion however that for reasons given later, the output figures for any sector in an input–output table need not necessarily be the same as the official output figures for that sector. Rows (2) to (9) show how the outputs of the other industries and services were allocated as between intermediate and final demand.

If we look now at the first *column* of Table 1.1 we see the various items which were purchased by the agricultural sector in 1960. Apart from the 2·180 mentioned above for Row (1) these were animal feed to the value of 13·860 purchased from the Food Processing sector, malt combings and brewers' grains for animal feed to the value of 0·496 purchased from the Drink and Tobacco

Table 1.1 Nine-Sector Input–Output Table for Ireland 1960

£ million

OUTPUT → / INPUT ↓	Intermediate						
	Agriculture, Forestry, Fishing	Food Processing	Drink, Tobacco	Textiles, Apparel	Metals, Engineering	All Other Manufact. and Mining	Construction
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
						I	
1. Agriculture, Forestry, Fishing	2·180	77·000	3·104	0·550	-	1·033	-
2. Food Processing	13·860	19·345	0·200	-	-	1·995	-
3. Drink, Tobacco	0·496	0·204	2·108	-	-	0·119	-
4. Textiles, Apparel	0·400	1·047	0·007	13·015	-	0·979	-
5. Metals, Engineering	2·942	0·505	0.095	0·120	2·537	0·779	4·545
6. All Other Manufact. and Mining	9·314	6·834	1·707	3·497	2·106	9·537	12·261
7. Construction	0·150	0·464	0·235	0·180	0·127	0·440	2·944
8. Electricity, Gas, Water	0·547	1·129	0·157	0·611	0·613	1·884	0·355
9. Services	11·020	13·658	0·832	5·200	2·819	4·925	4·590
10. Total Inter-Industry	40·909	120·186	8·445	23·173	8·202	21·691	24·695
Primary Inputs						III	
11. Imports	15·294	15·418	6·168	21·329	27·203	39·723	6·227
12. Indirect Taxes	11·559	1·540	42·550	0·303	2·800	0·873	0·726
13. Less Subsidies	−7·317	−3·337	-	−0·100	−0·149	−2·262	-
14. Wages, Salaries, Profits etc.	133·600	20·609	12·649	19·557	17·023	34·198	34·325
15. Depreciation	6·300	1·900	1·800	1·100	1·000	3·000	0·800
16. Total GNP Arising	144·142	20·712	56·999	20·860	20·674	35·809	35·851
17. Total Primary Inputs	159·436	36·130	63·167	42·189	47·877	75·532	42·078
18. Total Input = Total Output	200·345	156·316	71·612	65·362	56·079	97·223	66·773

£ million

Demand			Final Demand					
			Consumption					
Electricity, Gas, Water	Services	Total Intermediate	Household	Govern-ment	Capital Formation	Exports	Total Final	Total Output = Total Input
(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)
					II			
-	1·143	85·010	62·111	0·803	2·671	49·750	115·335	200·345
-	0·105	35·505	78·712	0·014	−1·009	43·094	120·811	156·316
-	0·100	3·027	55·834	-	+0·040	12·711	68·585	71·612
-	1·078	16·526	25·841	-	1·995	21·000	48·836	65·362
1·981	4·561	18·065	16·453	0·250	13·561	7·750	38·014	56·079
3·022	10·212	58·490	16·628	0·686	2·755	18·664	38·733	97·223
0·188	6·382	11·110	2·036	13·305	40·322	-	55·663	66·773
0·164	3·019	8·479	11·582	0·566	4·068	0·059	16·275	24·754
0·218	19·487	62·749	139·195	50·849	6·428	42·090	238·562	301·311
5·573	46·087	298·961	408·392	66·473	70·831	195·118	740·814	1 039·775
					IV			
3·774	7·855	142·991	62·295	1·764	25·983	3·345	93·387	236·378
0·465	9·200	70·016	31·884	-	0·886	3·175	35·945	105·961
-	−6·701	−19·866	-	-	-	-	-	−19·866
12·042	229·570	513·573	-	-	-	33·912	33·912	547·485
2·900	15·300	34·100	-	2·500	−1·100	-	1·400	35·500
15·407	247·369	597·823	31·884	2·500	−0·214	37·087	71·257	669·080
19·181	255·224	740·814	94·179	4·264	25·769	40·432	164·644	905·458
24·754	301·311	1 039·775	502·571	70·737	96·600	235·550	905·458	1 945·233

sector; ropes, sacks etc. to the value of 0·400 from Textiles and Apparel, and machinery, repairs and parts of the value of 2·942 from the Metals and Engineering industry. Farmers paid 9·314 for lime, fertilisers and chemicals to All Other Manufacturing and Mining; 0·150 to the Construction industry; 0·547 for Electricity, Gas and Water, and 11·020 for veterinary and other services. The total of all these items which came from industries within the State including that which agriculture purchased from itself was 40·909.

The next item in the column is imports valued at 15·294 such as live animals, oil cakes, compound fertilisers, seeds, fuel oil etc. Unprocessed feeds such as unmilled cereals and raw fertilisers such as rock phosphate are not included in this entry. These items are assumed to be imported by the industries using them as raw materials, which industries process them for sale to the agricultural etc. sector.

The other entries in the Agriculture column are: Indirect Taxes (11·559), Subsidies (—7·317), Wages, Salaries, Profits, etc. (133·600), Depreciation (6·300), and GNP Arising (144·142). Indirect taxes are rates on land and farm buildings together with customs and excise duties on farm inputs. Items like petrol and fuel oil used in farming are included in the import row at the price before customs duty, while the duty is entered in the indirect tax row. Subsidies are entered with a negative sign since they are a receipt and not an expense. The figure for GNP Arising is obtained by deducting imports from total primary inputs, i.e. 159·436—15·294 = 144·142. It may also be obtained by adding indirect taxes less subsidies plus depreciation to wages, salaries, profits etc., i.e. 133·600 + 11·559—7·317 + 6·300 = 144·142. This result derives from national accounting definitions and so it can be seen that the input—output system is intimately associated with the national accounts.

It should be noted that the total of the Agriculture column is exactly the same as that of the corresponding row. This is a feature of the input—output system, which is discussed later.

It would be tedious at this stage to discuss the entries in the other rows and columns of the table but sufficient has been said to show in a general way how the various figures are derived. Reference will be made later to a number of other entries.

Essential Features of a Transactions Table

As can be seen, Table 1.1 is divided vertically by a bold line into two parts. The part on the left represents the inputs to the production processes of the productive sectors and that on the right the sales to the final disposal sectors. Each of these parts is further subdivided horizontally into two sections so as to distinguish between what are called intermediate, and primary, inputs. The table therefore contains four quadrants.

The first quadrant shows the flows of goods and services which are both

produced and consumed in the process of current production. These are usually referred to as inter-industry flows or Intermediate Demand. The second quadrant shows the various elements of Final Demand for the output of each producing sector. In Table 1.1 Final Demand is shown as consisting of Household and Government consumption, and Capital Formation which includes stock changes and exports (visible and invisible). The third quadrant shows what are called Primary Inputs, to the productive sectors. These inputs are described as primary because they are not part of the output of current production, as defined by the rows forming quadrants I and II. The fourth quadrant shows the Primary Inputs which go directly to the Final Demand sectors.

An essential feature of the Transactions Table is that there must be the same number of rows in quadrant I as there are columns. In other words this quadrant must always be a square matrix, a matrix, as explained in the Mathematical Appendix, being an array of figures set out in rows and columns. The same restriction is not imposed on any of the other quadrants and in practice the number of rows in these quadrants is seldom equal to the number of columns. A schematic layout of a transactions table is as follows:

	1 .... n	1 .... m	
1 : : n	I (n × n)	II (n × m)	1 : : n
1 : : p	III (p × n)	IV (p × m)	1 : : p
	1 .... n	1 .... m	

Reference to Table 1.1 shows that the first quadrant consists of nine rows and a similar number of columns. It is for this reason that it is called a 9 × 9 table. In the tenth row are given the totals of the other nine rows, and similarly for the entries in the tenth column. The inclusion of a row and a column for such totals is not an essential feature of a transactions table and these entries are sometimes omitted in order to save printing space.

The second quadrant of Table 1.1 consists of nine rows and four columns together with a row and column for totals. As in the case of quadrant I the latter entries are not an essential feature of a transactions table, but a total column is practically always included in constructing such a table since it is usually required in subsequent analyses of the entries.

The third quadrant consists of five rows and nine columns. In addition a row and column are given for totals and a further row for GNP (Gross National Product) Arising. The latter rows and columns are not essential and may be omitted if it is necessary to save space.

In constructing Table 1.1 all the imports have been included with the primary

inputs but as will be shown later (Chapter 4) certain imports may more properly be entered elsewhere in the table. In addition to imports, other primary inputs specified in Table 1.1 are: (a) Indirect Taxes, (b) Subsidies, (c) Wages, Salaries, Profits etc. and (d) Depreciation.

Of these primary inputs, wages, salaries, profits and depreciation are sometimes referred to as "factor" inputs because they relate to what older economists called the factors of production, i.e. land, labour, and capital. Wages and salaries are the payments or returns to labour; profits and depreciation are the returns to capital, and rent which is usually included in profit is the return or charge for land. In modern times the use of the term factor in relation to inputs may possibly lead to confusion. Present-day economists lay little emphasis on the traditional factors of production, and readers may well ask why land is a factor of production while material inputs, like fertilisers, animal feed, chemicals, pesticides, etc. are classed as non-factor inputs. For this reason the terms factor and non-factor in relation to inputs should if possible be avoided.

The fourth quadrant proper consists of five rows and four columns. In addition, as in quadrant III, there is a row for Total GNP Arising. This quadrant is sometimes omitted from transactions tables but this omission is not to be recommended since without the entires in quadrant IV the input–output system cannot be fully reconciled with the national accounts. Full integration of the two systems is most important since it improves the quality of both and laces together the whole economy as no other procedure can. The following entries are recorded in quadrant IV of Table 1.1.

Imports

The figures recorded along the Import row in quadrant IV are the imports which go directly for household and government consumption, for capital formation and for re-export. They include such invisibles as expenditure by residents of the country on travel abroad. The total value of these imports given in Row (11) of Column (15) is 93·387.

Indirect Taxes

The figures in the Household and Capital Formation columns of this row are mainly rates on dwellings and customs duties on imports. Customs and Excise duties on drink, tobacco etc. manufactured in the State are not entered in quadrant IV but are included in the indirect taxes row of the sectors producing these goods, e.g. the entry of 42·550 in Row (12) of Column (3) represents the customs and excise duties on drink and tobacco manufactured or processed in the State. The figure of 3·175 in the Exports column of the Indirect Taxes row is the estimated amount of harbour dues received from foreigners and of indirect taxes on tourist purchases in 1960. The total value of indirect taxes in final demand given in Column (15) is 35·945. The total for all sectors

given in Column (16) is 105·961. Hence the amount allocated to intermediate sectors is 70·016 (i.e. the entry in Column (10)).

Subsidies

The total value of subsidies in 1960 as given in Column (16) of the Subsidy row is −19·866. None of the subsidies are however allocated to final demand; all go to intermediate sectors. It might be mentioned however, that direct subsidies on exported goods may be entered in the Export column of the Subsidy row.

Wages, Salaries and Profits

The total value of wages, salaries and profits accruing to persons in the State in 1960 as given in Column (16) of Row (14) was 547·485. As might be expected, most of the wages and profits are allocated to the intermediate sectors (i.e. 513·573) leaving only 33·912 for Final Demand. This latter amount, which is entered in the Export column of Row (14) represents profits, pensions and emigrants' remittances received from abroad less profits etc. paid out to foreigners. The amount of profits received from abroad in 1960 was 51·902 and the amount paid out to foreigners was 17·990 giving the net figure of 33·912.

Depreciation

The total amount set aside for depreciation in all sectors of the economy in 1960 as given in Column (16) of Row (15) was 35·500. As can be seen, most of this is allocated to the intermediate sectors, i.e. 34·100. Of the remainder, 2·500 is allocated to the Government Consumption column, offset by −1·100 in the Capital Formation column. The latter figure is the allowance made for stock appreciation in preparing the national accounts.

Total GNP Arising

The total value of GNP at market prices arising in the economy in 1960 was 669·080.* This figure is given in Column (16) of Row (16). The value of GNP arising in the intermediate sectors is 597·823 and in the final sectors (in quadrant IV) is 71·257. As stated previously, figures for GNP are obtained by deducting from the total primary inputs of any sector the imports of that sector. For the economy as a whole the calculation is similar. Thus total primary inputs for all sectors given in Row (17) of Column (16) is 905·458. Total imports in Row (11) of Column (16) are 236·378 and when the latter figure is deducted from the former we obtain 669·080 which is the total value of GNP at market prices in 1960. The overall value of GNP may also be obtained by ded-

* This is the figure for GNP at market prices published in the 1961 issue of *National Income and Expenditure* issued by the Central Statistics Office. The figure has subsequently been revised as a result of more up-to-date information becoming available.

ucting imports from the total of all the final demand columns, since an essential feature of the table is that total final demand is equal to total primary inputs, i.e. Row (18) of Col. (15) = Row (17) of Col. (16) = 905·458.

Equality of Rows and Columns

In an input–output table the total value of output of each productive sector, i.e. the row total, is always equal to its total expenditure on inputs, i.e. the column total. No such equality, however, is imposed on the Final Demand sectors or on the Primary Input sectors. It is sufficient that all the final sectors taken together should be equal to the total of the primary inputs. In line with these equality restrictions it should be noted from Table 1.1 that the totals of each of the first nine rows are equal to those of each of the first nine columns. Also the sum of all the Final Demand rows in quadrant II, i.e. the figure of 740·814 in the 15th column of the 10th row, is equal to the sum of the Primary Input rows in quadrant III, i.e. the figure in the 10th column of the 17th row. This result occurs because the inter-industry block has been deducted from the aggregates of the 9 rows across and the nine columns down, these two aggregates being equal, i.e. 1 039·775.

The equality of inputs and outputs in a transactions table is of course an accounting identity. In preparing the accounts for each sector, total output is first determined. Expenses, including imported inputs and depreciation, are deducted from this and the balance is defined as "income arising" in the sector. This latter item can be taken as being equivalent to Wages, Salaries, and Profits in Table 1.1. Hence for each production sector Wages, Salaries, and Profits plus other expenses are equal to output. Figures for sectoral outputs in any year, however, may not be the same in national statistical publications as in input–output tables. This difference occurs for the following reasons.

In preparing national statistics, items produced in a sector for further consumption in the same sector may be netted out. In preparing input–output tables on the other hand intra-sectoral transactions (sales by a sector to itself) of this kind are usually not netted out and appear as entries in the cells of the principal diagonal of the tables (Row 1 of Column 1, Row 2 of Column 2 etc.). Hence the sectoral outputs in input–output tables are usually greater than the official output figures. Inputs are of course correspondingly greater also so that the input–output equality is preserved. Some input–output tables exclude intrasectoral transactions and in such cases all the cells on the principal diagonal are blank. It should be kept in mind also that sectors may be defined differently in input–output tables and national accounts, thus making it difficult for readers to reconcile the output figures for a particular sector in the two sets of tables.

United Kingdom Input–Output Table for 1963

Table 1.2 gives a 9-sector input–output table for the United Kingdom for

the year 1963, obtained by aggregating a 70-sector table compiled by the Central Statistical Office, London*. The layout corresponds to that of Table 1.1 except for a row of Primary Inputs denoted "Sales by Final Buyers", which has a row aggregate of zero. The entries in this row relate to second-hand transactions within the economy and not to outputs of the productive sectors during 1963, hence all purchases must come from reductions in stocks elsewhere. Thus the positive entries for purchases are balanced by negative entries for sales, mainly in the Government and Capital Formation columns. In the case of the Irish table such transactions are relatively trivial and are submerged in Row 9, Services.

Table 1.2 has indirect Taxes less Subsidies combined in the single Row (13), and Depreciation is included in Profits, which with Wages and Salaries form Row (14). As can be seen, about four-fifths of total indirect taxes less subsidies appear in the Final Demand section of Row (13). This indicates that the major taxes such as those on drink and tobacco are charged to households rather than in the producing sector, i.e. Column (3). The opposite procedure is adopted in the Irish table, the taxes being charged in the producing rather than the consuming sectors. Both these procedures are legitimate.

Comparison of the main features of Table 1.2 for the U.K. with that of Ireland given in Table 1.1 is an interesting exercise for the reader, and only a few hints will be given here. There is first of all the vast increase in scale for the heavily industrialized U.K. economy relating to some 50 million population, compared with the Irish economy, for some 3 million population. Household Expenditure in 1963 for the U.K. is £20 125 million as against £503 million for Ireland in 1960, roughly 40 times as large as the latter. U.K. Exports are roughly 20 times as large and Capital Formation some 50 times greater. The proportions between the outputs of the productive sectors of the U.K. are different to those in the Irish table, only part of the difference being due to the alternate treatments of Indirect Taxes. Output of U.K. Agriculture is less than 4 per cent of the aggregate output of all productive sectors, whereas Table 1.1 shows this Irish sector to form nearly 20 per cent of aggregate output, which was £1 040 million. U.K. Manufacturing other than Food, Drink and Tobacco (sectors (4) to (6) inclusive) forms about 40 per cent of the output of all productive sectors (Row (10), Column (16), value £50 430 million) as against roughly 20 per cent for Ireland (sectors (4) to (6) inclusive).

It is thus evident that even Transactions tables alone provide a wealth of information for comparisons between economies of different countries, at various or similar stages of economic development, apart from the interrelations which they show for the structure of a single economy.

* For exact source see footnote to Table 1.2.

Table 1.2 Nine-Sector Input–Output Table for the United Kingdom, 1963

£ million

	Intermediate					
Outputs → / Inputs ↓	Agriculture etc. (1)	Mining, Quarrying (2)	Food, Drink, Tobacco (3)	Chem., Metals, Vehicles (4)	Textiles, Paper, Printing (5)	Other Manufacturing (6)
1. Agriculture, Forestry, Fishing	277	-	398	3	33	9
2. Mining, Quarrying	5	5	16	225	40	62
3. Food, Drink, Tobacco	324	1	460	25	1	1
4. Chemicals, Metals, Vehicles	177	127	237	4 913	252	288
5. Textiles, Paper, Printing	11	21	121	197	410	132
6. Other Manufacturing	20	39	51	387	58	182
7. Construction	30	21	13	59	12	6
8. Gas, Electricity, Water	20	34	28	261	49	48
9. Services	236	86	380	1 360	484	257
10. Total Inter-Industry	1 100	334	1 704	7 430	1 339	985
Primary Inputs						
11. Imports	133	11	606	1 277	603	235
12. Sales by Final Buyers*	3	-	−1	110	9	4
13. Indirect Taxes less Subsidies	−246	117	57	174	69	50
14. Wages & Salaries, Profits	954	733	965	5 320	1 725	931
15. Total Primary Inputs	844	761	1 627	6 881	2 406	1 220
16. Total Input = Total Output	1 944	1 095	3 331	14 311	3 745	2 205

* Sales by Final Buyers represents sales and purchases of second-hand goods and scrap metal etc.
Source: Table D of "Input–Output Tables for the United Kingdom 1963", compiled by Central Statistical Office and published by H.M.S.O. London 1970.

£ million

Demand				Final Demand					
Construct-ion	Gas, Elec-tricity, Water	Serv-ices	Total	Consumption		Capital form-ation	Exports	Total Final	Total Output =Total Input
				House-holds	Govern-ment				
(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)
-	1	14	735	1 022	101	35	51	1 209	1 944
65	346	41	805	220	32	−13	51	290	1 095
-	-	32	844	2 176	37	48	226	2 487	3 331
407	163	739	7 303	1 203	1 046	1 900	2 859	7 008	14 311
13	6	601	1 512	1 497	130	31	575	2 233	3 745
629	16	150	1 532	350	51	70	202	673	2 205
657	4	112	914	389	291	2 294	15	2 989	3 903
11	49	209	709	670	82	167	6	925	1 634
218	130	1 572	4 723	8 164	3 493	430	1 452	13 539	18 262
2 000	715	3 470	19 077	15 691	5 263	4 962	5 437	31 353	50 430
95	17	676	3 653	1 586	184	250	273	2 293	5 946
12	-	42	179	263	−353	−177	88	−179	-
65	67	442	695	2 585	90	100	17	2 792	3 487
1 731	835	13 632	26 826	-	-	-	-	-	26 826
1 903	919	14 792	31 353	4 434	−79	173	378	4 906	36 259
3 903	1 634	18 262	50 430	20 125	5 184	5 135	5 815	36 259	86 689

REGIONAL INPUT–OUTPUT MODELS AND MODELS EMPHASIZING SOME SPECIFIC INDUSTRY

Regional Models

In addition to the preparation of national input–output tables the construction of regional models is a popular feature of the work in many research institutes, particularly in the USA where adequate statistical data are available on a regional or State basis [1,2,3,4]. In the preparation of such models purchases from other regions are treated as if they were imports, and sales to other regions are treated as exports. The dependence of a region on its own and on outside resources is therefore apparent from the table.

Regional models have many advantages for policy makers but they are of particular importance in showing how policies which may be of overall benefit to the nation may work to the detriment of a particular region within the State. For example a policy of protecting growers of feed grains which may be of advantage to the economy as a whole may, through its effect on feed prices, work to the disadvantage of pig and poultry producers in non-grain-growing regions. Similarly a policy of subsidizing the production of calves from beef cows, which may be of national benefit, may have the effect of reducing the demand for dairy calves and so be detrimental to farmers in dairying regions, and so on. Indeed the construction of regional models may in many ways be a more important exercise than the preparation of country tables. At the time of writing, models for particular regions in Ireland have not been prepared but it is hoped that this deficiency will be made good shortly.

Models Emphasizing some Specific Industry

The construction of models emphasizing specific industries is now a popular feature of research in many countries[5] and tables emphasizing the agricultural sector have been prepared for Ireland by Attwood,[6] and by O'Connor and Breslin.[7] The essential feature of these models is that the industry being emphasized is disaggregated into a number of sub-sectors, and cost structures for these worked out. Each sub-sector then becomes a sector in its own right and is included as a row and column in the inter-industry quadrant of the transactions table. Industries closely connected with the industry under review are usually also included in the inter-industry quadrant of the table, but not in such highly disaggregated form. Industries and services not closely connected with the industry under review are usually aggregated into a single sector and may or may not be included in the inter-industry quadrant. If they are included in the latter quadrant their cost structures must be worked out, which operation can add considerably to the calculations. They may however be entered in a column headed "other industries" in the final demand section of the table, a procedure which obviates the necessity for calculating cost structures and re-

duces considerably the volume of work involved. The procedure to be adopted will of course depend on the use for which the table is required, but as a general rule it is not necessary to work out cost structures for all industries in preparing such tables.

One of the main uses of a model emphasizing a particular industry is to study the inter-relationships between sub-sectors of the industry concerned. Such a study is of great benefit for planning purposes as it enables administrators to devise consistent plans and to determine the resource requirements for the planned production. Without the help of such a table administrators may make plans which are internally inconsistent, which are unrealistic regarding imported resources and which may have serious repercussions on the balance of payments.

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2 TECHNICAL AND INTERDEPENDENCE COEFFICIENTS

Technical Coefficients

After preparing the transactions table, which is the statistical basis of the input–output system, the next operation is to calculate what is called the unit cost structure or the technical coefficients. Though Table 1.1 is small in comparison with the national table from which it was derived it is still rather cumbersome for showing the methods of calculating technical and other coefficients. For this purpose Table 2.1 has been prepared by aggregating all the industrial sectors in Table 1.1 to give a small model having only three inter-industry rows and columns. Technical coefficients calculated from this table are shown in Table 2.2. These coefficients are calculated by dividing every item in quadrants I and III of Table 2.1 by the total of the column in which the item is recorded. For example, the internal flow within Agriculture, 2·180, when divided by 200·345 (the total of the agricultural column), gives 0·0109, the entry in Row 1 of Column 1 of Table 2.2. Similarly the sales of agricultural produce to industry, 81·687, when divided by 538·119 (the total of the industry column) gives 0·1518, the entry in the first row of Column (2) of the table, and so on for the other items. The ratios or technical coefficients so calculated are interpreted as shown in Table 2.2.

Each £1 of agricultural output requires £0·0109 of inputs from within its own sector, materials to the value £0·1383 from industry, services to the value of £0·0550, and total primary inputs to the value of £0·7958. Similarly every £1 of output from industry uses agricultural produce to the value of £0·1518, industrial produce to the value of £0·1822, services valued at £0·0599, and total primary inputs valued at £0·6061. The coefficients for the services sector have similar meanings.

A highly aggregated U.K. Transactions table derived from Table 1.2 is given as Table 2.3 and technical coefficients derived from this table are given in Table 2.4. Similar highly aggregated Netherlands tables are given as an Appendix to this Chapter.

Interdependence Coefficients

Because of the inter-relationship between different sectors of an economy, a change in the final demand for the products of one sector causes ramifications throughout the system which change not only the outputs of the sector concerned but also those of most or perhaps all of the other sectors of the economy.

TABLE 2.1 Highly Aggregated Input–Output Table for Ireland, 1960

£ Million

Inputs ↓ / Outputs →	Inter-Industry				Final Demand					
	Agriculture	Industry	Services	Total Inter-Industry	Consumption		Capital Formation	Exports	Total Final Demand	Output
					Household	Govt.				
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
	I				II					
1. Agriculture	2·180	81·687	1·143	85·010	62·111	0·803	2·671	49·750	115·335	200·345
2. Industry	27·709	98·036	25·457	151·202	207·086	14·821	61·732	103·278	386·917	538·119
3. Services	11·020	32·242	19·487	62·749	139·195	50·849	6·428	42·090	238·562	301·311
4. Total Inter-Industry	40·909	211·965	46·087	298·961	408·392	66·473	70·831	195·118	740·814	1 039·775
Primary Inputs	III				IV					
5. Imports	15·294	119·842	7·855	142·991	62·295	1·764	25·983	3·345	93·387	236·378
6. Indirect Taxes	11·559	49·257	9·200	70·016	31·884	-	0·886	3·175	35·945	105·961
7. Subsidies	−7·317	−5·848	−6·701	−19·866	-	-	-	-	-	−19·866
8. Wages, Salaries, Profits, etc.	133·600	150·403	229·570	513·573	-	-	-	33·912	33·912	547·485
9. Depreciation	6·300	12·500	15·300	34·100	-	2·500	−1·100	-	1·400	35·500
10. Total Primary Inputs	159·436	326·154	255·224	740·814	94·179	4·264	25·769	40·432	164·644	905·458
11. Input = Output	200·345	538·119	301·311	1 039·775	502·571	70·737	96·600	235·550	905·458	1 945·233

TABLE 2.2 Technical Coefficients (Ireland 1960) (Based on Table 2.1)

Inputs ↓	Agriculture	Industry	Services
1. Agriculture	0·0109	0·1518	0·0038
2. Industry	0·1383	0·1822	0·0845
3. Services	0·0550	0·0599	0·0647
4. Total Inter-Industry	0·2042	0·3939	0·1530
Primary Inputs			
5. Imports	0·0763	0·2227	0·0261
6. Indirect Taxes	0·0577	0·0915	0·0305
7. Subsidies	−0·0365	−0·0109	−0·0222
8. Wages, Salaries, Profits	0·6668	0·2795	0·7619
9. Depreciation	0·0314	0·0232	0·0508
10. Total Primary Inputs	0·7958	0·6061	0·8470
11. Total Inputs	1·0000	1·0000	1·0000

Note: Because of rounding errors, the sum of the individual entries in a column may not add exactly to unity, nor to the subtotals shown in Rows (4) and (10), which were calculated from the actual transactions.

One of the main aims of input—output analysis is to study these changes but unfortunately the technical coefficients of Table 2.2 cannot be used directly for this purpose as they show only what are known as the direct or first order effects of changes in final demand. To study second and higher-order effects, other operators known as total or interdependence coefficients are required.

To understand how these coefficients are derived recourse must be had to algebra and in order to develop the required equations it is necessary to replace some of the figures in Table 2.1 and 2.2 by appropriate symbols. The relevant commodity flows of Table 2.1 are given in the form of symbols in Table 2.5.

In Table 2.5 the total outputs of agriculture, industry, and services are represented by X_1, X_2 and X_3 respectively. The final demands for these sectors are represented by Y_1, Y_2 and Y_3 respectively while x_{11}, x_{12}, x_{13}, x_{21} etc. (referred to mathematically as the x_{ij}'s) are used to represent the internal flows

TABLE 2.3 Highly Aggregated Input–Output Table for the United Kingdom, 1963

£ Million

Inputs ↓ / Outputs →	Inter-Industry				Final Demand				
	Agric-ulture	Industry	Services	Total Inter-Industry	Con-sumption	Capital form-ation	Exports	Total Final Demand	Output
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
1. Agriculture	277	444	14	735	1 123	35	51	1 209	1 944
2. Industry	587	11 148	1 844	13 619	8 174	4 497	3 934	16 605	30 224
3. Services	236	2 915	1 572	4 723	11 657	430	1 452	13 539	18 262
4. Total Inter-Industry	1 100	14 507	3 470	19 077	20 954	4 962	5 437	31 353	50 430
Primary Inputs									
5. Imports	133	2 844	676	3 653	1 770	250	273	2 293	5 946
6. Sales by final buyers	3	134	42	179	−90	−177	88	−179	-
7. Indirect taxes less subsidies	−246	499	442	695	2 675	100	17	2 792	3 487
8. Wages, Salaries, Profits	954	12 240	13 632	26 826	-	-	-	-	26 826
9. Total Primary Inputs	844	15 717	14 792	31 353	4 355	173	378	4 906	36 259
10. Total Inputs	1 944	30 224	18 262	50 430	25 309	5 135	5 815	36 259	86 689

TABLE 2.4 Technical Coefficients (U.K. 1963) (Based on Table 2.3)

Inputs ↓	Agriculture	Industry	Services
1. Agriculture	0·1425	0·0147	0·0008
2. Industry	0·3020	0·3688	0·1032
3. Services	0·1214	0·0964	0·0861
4. Total Inter-Industry	0·5658	0·4800	0·1900
Primary Inputs			
5. Imports	0·0684	0·0941	0·0370
6. Sales by Final Buyers	0·0015	0·0044	0·0023
7. Indirect Taxes less Subsidies	−0·1265	0·0165	0·0242
8. Wages, Salaries, Profits	0·4907	0·4050	0·7465
9. Total Primary Inputs	0·4342	0·5200	0·8100
10. Total Inputs	1·0000	1·0000	1·0000

Note: **Because of rounding errors, the sum of the individual entries in a column may not add exactly to unity, nor to the subtotals for Rows (4) and (9), which are calculated from the actual transactions.**

TABLE 2.5 Commodity Flows by Sector of Origin and Destination in Symbolic Terms

Inputs	Intermediate Demand			Total Final Demand	Total Output
	(1)	(2)	(3)		
Agriculture 1	x_{11}	x_{12}	x_{13}	Y_1	X_1
Industry 2	x_{21}	x_{22}	x_{23}	Y_2	X_2
Services 3	x_{31}	x_{32}	x_{33}	Y_3	X_3
All primary inputs	Z_1	Z_2	Z_3	-	-
Total inputs	X_1	X_2	X_3		

within the economy. Total primary inputs are represented by Z_1, Z_2, and Z_3. For the sake of brevity symbols for the individual primary inputs are not given but such symbols could be given if required.

The various flows in Table 2.5 may be represented by the following system of linear equations which are used later in deriving the required results.

$$X_1 = x_{11} + x_{12} + x_{13} + Y_1$$
$$X_2 = x_{21} + x_{22} + x_{23} + Y_2 \qquad (2.1)$$
$$X_3 = x_{31} + x_{32} + x_{33} + Y_3$$

The inter-industry technical coefficients in Table 2.2 are given in the form of symbols in Table 2.6. In input–output terminology this table is usually referred to as the A matrix, though it could of course be referred to by any other letter such as B, C, D, etc.

TABLE 2.6 Inter-Industry Technical Coefficients in Symbolic Form (A Matrix)

Sector	Intermediate Demand		
	Agriculture	Industry	Services
	(1)	(2)	(3)
Agriculture 1	a_{11}	a_{12}	a_{13}
Industry 2	a_{21}	a_{22}	a_{23}
Services 3	a_{31}	a_{32}	a_{33}

As explained above the technical coefficients are calculated by dividing the figures in quadrants I and III of the Transactions table by the corresponding column totals,* therefore from Tables 2.5 and 2.6:

$$a_{11} = \frac{x_{11}}{X_1}; \quad a_{12} = \frac{x_{12}}{X_2}; \quad a_{13} = \frac{x_{13}}{X_3}; \quad a_{21} = \frac{x_{21}}{X_1}$$

or in general

$$a_{ij} = \frac{x_{ij}}{X_j} \qquad (2.2)$$

where i represents the number of the row and j the number of the column in which a coefficient is located.

It follows from (2.2) that

$$x_{ij} = a_{ij}X_j \qquad (2.3)$$

so that

$$x_{11} = a_{11}X_1; \quad x_{12} = a_{12}X_2; \quad x_{13} = a_{13}X_3; \quad x_{21} = a_{21}X_1$$

and so on.

Substituting equation (2.3) into equation (2.1) we obtain:

$$X_1 = a_{11}X_1 + a_{12}X_2 + a_{13}X_3 + Y_1$$

* It is very important to keep in mind that we divide *each column* and *not each row* by its total input figure.

$$\begin{aligned} X_2 &= a_{21}X_1 + a_{22}X_2 + a_{23}X_3 + Y_2 \\ X_3 &= a_{31}X_1 + a_{32}X_2 + a_{33}X_3 + Y_3 \end{aligned} \tag{2.4}$$

Transferring all the X's to the left-hand side and re-grouping we obtain:

$$\begin{aligned} (1-a_{11})X_1 - a_{12}X_2 - a_{13}X_3 &= Y_1 \\ -a_{21}X_1 + (1-a_{22})X_2 - a_{23}X_3 &= Y_2 \\ -a_{31}X_1 - a_{32}X_2 + (1-a_{33})X_3 &= Y_3 \end{aligned} \tag{2.5}$$

Substituting the technical coefficients from Table 2.2 for the a_{ij}'s in equation (2.5) we obtain:

$$\begin{aligned} (1-0{\cdot}0109)X_1 - 0{\cdot}1518X_2 - 0{\cdot}0038X_3 &= Y_1 \\ -0{\cdot}1383X_1 + (1-0{\cdot}1822)X_2 - 0{\cdot}0845X_3 &= Y_2 \\ -0{\cdot}0550X_1 - 0{\cdot}0599X_2 + (1-0{\cdot}0647)X_3 &= Y_3 \end{aligned} \tag{2.6}$$

The interdependence coefficients are obtained from (2.6) by first simplifying the diagonal elements and then expressing the X's in terms of the Y's. The system with simplified diagonal elements is

$$\begin{aligned} 0{\cdot}9891X_1 - 0{\cdot}1518X_2 - 0{\cdot}0038X_3 &= Y_1 \\ -0{\cdot}1383X_1 + 0{\cdot}8178X_2 - 0{\cdot}0845X_3 &= Y_2 \\ -0{\cdot}0550X_1 - 0{\cdot}0599X_2 + 0{\cdot}9353X_3 &= Y_3 \end{aligned} \tag{2.7}$$

Either by basic algebra (solving simultaneous linear equations (2.7) for X_1, X_2, X_3) or by matrix inversion shown in the Mathematical Appendix to this book (equations (a.28) to (a.31)), the solution of (2.7) is given by:

$$\begin{aligned} X_1 &= 1{\cdot}0394Y_1 + 0{\cdot}1945Y_2 + 0{\cdot}0218Y_3 \\ X_2 &= 0{\cdot}1833Y_1 + 1{\cdot}2652Y_2 + 0{\cdot}1150Y_3 \\ X_3 &= 0{\cdot}0729Y_1 + 0{\cdot}0925Y_2 + 1{\cdot}0778Y_3 \end{aligned} \tag{2.8}$$

The coefficients of Y_1, Y_2 and Y_3 are the interdependence coefficients. Even for a small model such as this a large number of calculations is required to obtain these coefficients. To understand how calculations are made some elementary notions of matrix algebra are required.

The matrix manipulations required for the simple model given here are explained below. Equation system (2.5) above can be written in matrix form as*

$$\begin{bmatrix} (1-a_{11}) & -a_{12} & -a_{13} \\ -a_{21} & (1-a_{22}) & -a_{23} \\ -a_{31} & -a_{32} & (1-a_{33}) \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} \tag{2.9}$$

* For an explanation of this notation see under "Typical input–output Matrices" in the Mathematical Appendix at the end of the book.

Here the box containing the a_{ij}'s is referred to as a 3 x 3 matrix, i.e. a matrix having 3 rows and 3 columns. For short it may be written* as $(I-A)$. The columns of X's and Y's are called vectors which may be written as X and Y respectively, while the whole system may be written in abbreviated matrix form:

$$(I-A)X = Y \tag{2.10}$$

Now in input–output analysis the vector of Y's, i.e. vector of final demand, is usually assumed to be exogenous or given, and the problem is to determine the vector of outputs, i.e. the X's. If (2.10) were an ordinary equation the value of X would be obtained by dividing Y by $(I-A)$, i.e. $X = Y/(I-A)$. As explained in the Mathematical Appendix, however, division in matrix algebra cannot be performed in the ordinary way. If we wish to divide one matrix by another we multiply the one by the reciprocal or inverse of the other. Thus the solution to (2.10) above is

$$X = (I-A)^{-1}Y \tag{2.11}$$

where $(I-A)^{-1}$ is the inverse of matrix $(I-A)$. The problem then is to determine the inverse of Matrix $(I-A)$. The numerical values for the elements of the $(I-A)$ matrix used in this example are the coefficients of the X's in (2.7) above.

$$(I-A) = \begin{bmatrix} 0{\cdot}9891 & -0{\cdot}1518 & -0{\cdot}0038 \\ -0{\cdot}1383 & 0{\cdot}8178 & -0{\cdot}0845 \\ -0{\cdot}0550 & -0{\cdot}0599 & 0{\cdot}9353 \end{bmatrix} \tag{2.12}$$

Its inverse, the derivation of which is shown in the Mathematical Appendix, is

$$(I-A)^{-1} = \begin{bmatrix} 1{\cdot}0394 & 0{\cdot}1945 & 0{\cdot}0218 \\ 0{\cdot}1833 & 1{\cdot}2652 & 0{\cdot}1150 \\ 0{\cdot}0729 & 0{\cdot}0925 & 1{\cdot}0778 \end{bmatrix} \tag{2.13}$$

It should be noted that the elements of (2.13) are the coefficients of the Y's in (2.8) above, i.e. the interdependence coefficients. Hence the interdependence coefficients are obtained by inverting the $(I-A)$ matrix of coefficients. The values shown in (2.8) and (2.13) are those obtained by a computer, rounded to 4 decimal places, and are more precise than those calculated in the Mathematical Appendix, equations (a.28) to (a.31).

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* For an explanation of this notation see under "Typical input–output Matrices" in the Mathematical Appendix.

Checking the Elements of the Inverse Matrix

It was stated above that one of the objects of input–output analysis is to determine the vector of outputs i.e. the X's, given the vector of final demands, i.e. the Y's. This is done as shown in equation (2.11), by multiplying the vector of Y's by the inverse of $(I - A)$, i.e. $X = (I - A)^{-1}Y$. In practical work we use this relationship also to check the elements of $(I - A)^{-1}$. To do this we take the figures for final demand in quadrant II of Table 2.1 and multiply them by the inverse matrix. If our arithmetic has been correct we should obtain (except for rounding errors) the vector of Total Outputs of Table 2.1 as follows,

$$\begin{bmatrix} 1{\cdot}0394 & 0{\cdot}1945 & 0{\cdot}0218 \\ 0{\cdot}1833 & 1{\cdot}2652 & 0{\cdot}1150 \\ 0{\cdot}0729 & 0{\cdot}0925 & 1{\cdot}0778 \end{bmatrix} \begin{bmatrix} 115{\cdot}335 \\ 386{\cdot}917 \\ 238{\cdot}562 \end{bmatrix} \simeq \begin{bmatrix} 200{\cdot}345 \\ 538{\cdot}119 \\ 301{\cdot}311 \end{bmatrix} \qquad (2.14)$$

where $\simeq$ means "approximately equal to".

In doing the actual calculations in (2.14) each element of Y is multiplied by a matching element of the first row of the matrix $(I - A)^{-1}$ and the products summed.* This sum should be approximately equal to 200·345, the first element in the X vector. The second element of the X vector, i.e. 538·119, is got by multiplying the elements of Y by matching elements of the second row of the matrix and summing the results, while in a similar manner the third element of the X vector, i.e. 301·311, is got by summing the products of the Y vector and the third row of the matrix. The calculations are as follows:

$$(1{\cdot}0394 \times 115{\cdot}335) + (0{\cdot}1945 \times 386{\cdot}917) + (0{\cdot}0218 \times 238{\cdot}562) = 200{\cdot}335$$
$$(0{\cdot}1833 \times 115{\cdot}335) + (1{\cdot}2652 \times 386{\cdot}917) + (0{\cdot}1150 \times 238{\cdot}562) = 538{\cdot}103$$
$$(0{\cdot}0729 \times 115{\cdot}335) + (0{\cdot}0925 \times 386{\cdot}917) + (1{\cdot}0778 \times 238{\cdot}562) = 301{\cdot}320$$

Except for rounding errors the figures on the right of the sign of equality check closely with the elements of the X vector in (2.14). Therefore our calculations are correct.

Interpretation of the Interdependence Coefficients

The interpretation of the interdependence coefficients, i.e. the elements of the inverse matrix, is explained briefly by reference to equation system (2.8) above. This system shows the relationship between the outputs of the three producing sectors, i.e. Agriculture, Industry, and Services, and the final demands for the products of these sectors. The first equation states that the output of Agriculture (X_1) is a function of (a) the final demand for agricultural goods (Y_1), (b) the final demand for industrial goods (Y_2), and (c) the final demand

* This is known technically as pre-multiplication of Y by $(I - A)^{-1}$ as explained in the Mathematical Appendix.

for services (Y_3). The other equations have similar meanings.

By looking at the columns of this equation system and keeping in mind the way in which the technical coefficients were derived we can say that for each £1 of final demand for agricultural produce (Y_1) the total output of agriculture is £1·0394, that of industry £0·1833 while that of services is £0·0729. This latter statement may not be easy to grasp at first, but if we refer back to equation (2.8) and think in terms of an increase of £1 in final demand for agricultural produce with no increase in the final demands for the products of the other sectors, then with $Y_1 = 1$ and $Y_2 = Y_3 = 0$, equation system (2.8) reduces to

$$\begin{aligned} X_1 &= 1{\cdot}0394 \\ X_2 &= 0{\cdot}1833 \\ X_3 &= 0{\cdot}0729 \end{aligned}$$

In a similar manner it can be shown that for each £1 of final demand for industrial goods (Y_2) the total output of agriculture is 0·1945, that of industry itself is 1·2652, while that of services is 0·0925. The outputs associated with a £1 increase in the final demand of the service sector can be calculated similarly.

It should be noted that for any sector the output required exceeds final demand because indirect relationships are expressed in the system. In fact the interdependence coefficients show both the direct and indirect effects of increasing final demand for any sector by one unit of value.

Examination of Direct and Indirect Effects of Changes in Demand

Though the total effects of demand changes are given by the interdependence coefficients, it is sometimes useful to break down these effects into their different components so as to isolate what are called first order, second order, third order etc. effects. An example explains how this is done.

Suppose that the final demands for the products of the agricultural sector are increased by one unit with no change in those of the other sectors. Now since the output of a sector is obtained by summing across the various row entries, an increase of one unit in final demand for agricultural produce must cause an immediate increase of one unit in agricultural output to supply this demand. The first column of Table 2.2 shows however that an increase of one unit in agricultural output requires:

(a) 0·0109 units of agricultural produce for internal transactions,
(b) 0·1383 units of industrial produce, and
(c) 0·0550 units of services.

These requirements will in turn increase the totals of their respective rows so that a one-unit increase in the demand for agricultural produce will, in addition to immediately increasing the output of agriculture by one unit, have the effect of increasing (a) the output of agriculture by a further 0·0109 units,

(b) the output of industry by 0·1383 units, and (c) the output of services by 0·0550 units. These are known as the first-order effects, and the amounts other than the unit required for direct final demand can be obtained by pre-multiplying the vector of final demand changes Y by the matrix of technical coefficients (A), i.e.

$$AY = X^{(1)} \tag{2.15}$$

where $X^{(1)}$ is the vector of first-order changes in output. The calculations written out in full are:

$$\begin{bmatrix} 0{\cdot}0109 & 0{\cdot}1518 & 0{\cdot}0038 \\ 0{\cdot}1383 & 0{\cdot}1822 & 0{\cdot}0845 \\ 0{\cdot}0550 & 0{\cdot}0599 & 0{\cdot}0647 \end{bmatrix} \begin{array}{c} Y \\ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \end{array} = \begin{array}{c} X^{(1)} \\ \begin{bmatrix} 0{\cdot}0109 \\ 0{\cdot}1383 \\ 0{\cdot}0550 \end{bmatrix} \end{array} \tag{2.16}$$

This result is obtained (as explained in the Mathematical Appendix at the back of the book) by pre-multiplying the vector of final demand Y in turn by each row of matrix A and summing the results as follows:

Agriculture: $(0{\cdot}0109 \times 1) + (0{\cdot}1518 \times 0) + (0{\cdot}0038 \times 0) = 0{\cdot}0109$
Industry: $(0{\cdot}1383 \times 1) + (0{\cdot}1822 \times 0) + (0{\cdot}0845 \times 0) = 0{\cdot}1383$
Services: $(0{\cdot}0550 \times 1) + (0{\cdot}0599 \times 0) + (0{\cdot}0647 \times 0) = 0{\cdot}0550$

Now the first-order effects give rise to second and higher-order effects because the first-order increases in output require further inputs to generate them and these in turn increase outputs further and so on.

Second-Order Effects

The second-order effects are obtained by pre-multiplying the vector of first-order effects by the technical coefficients in the same way as before, thus:

$$AX^{(1)} = X^{(2)} \tag{2.17}$$

where $X^{(1)}$ is the vector of first-order effects and
$X^{(2)}$ is the vector of second-order effects.
The figures written out in full are:

$$\begin{bmatrix} 0{\cdot}0109 & 0{\cdot}1518 & 0{\cdot}0038 \\ 0{\cdot}1383 & 0{\cdot}1822 & 0{\cdot}0845 \\ 0{\cdot}0550 & 0{\cdot}0599 & 0{\cdot}0647 \end{bmatrix} \begin{array}{c} X^{(1)} \\ \begin{bmatrix} 0{\cdot}0109 \\ 0{\cdot}1383 \\ 0{\cdot}0550 \end{bmatrix} \end{array} = \begin{array}{c} X^{(2)} \\ \begin{bmatrix} 0{\cdot}0213 \\ 0{\cdot}0314 \\ 0{\cdot}0124 \end{bmatrix} \end{array} \tag{2.18}$$

Third-Order Effects

The third-order effects are obtained by multiplying the vector of second-order effects by the matrix of technical coefficients in the same way as before.

Thus

$$AX^{(2)} = X^{(3)} \tag{2.19}$$

where $X^{(3)}$ is the vector of third-order effects.
The results of this multiplication are:

	$X^{(3)}$
Agriculture:	0·0050
Industry:	0·0097
Services:	0·0039

For the three sectors, the unit of final demand plus the total first-, second- and third-order effects are therefore:

	Unit of Final Demand	1st ORDER	2nd ORDER	3rd ORDER	
Agriculture:	1·000	+ 0·0109	+ 0·0213	+ 0·0050	= 1·0372
Industry:		0·1383	+ 0·0314	+ 0·0097	= 0·1794
Services:		0·0550	+ 0·0124	+ 0·0039	= 0·0713

It will be noted that with the exception of the agricultural sector the second-order effects are less than the first while in all cases the third-order effects are less than the second. The fourth-order effects, if calculated, would be less than the third, the fifth less than the fourth and so on until a point is reached when no further increase of any significance whatsoever is obtained for the sum of effects of all further orders. As can be seen, each order is obtained by multiplying the previous order by the matrix of technical coefficients, and the total effect of all orders by summing the various order effects.

Now since

$$\begin{aligned} X^{(1)} &= AY \\ X^{(2)} &= AX^{(1)} = A^2Y \\ X^{(3)} &= AX^{(2)} = A^3Y \\ X^{(n)} &= AX^{(n-1)} = A^nY \end{aligned}$$

the total of the various effects is

$$X^{(1)} + X^{(2)} + X^{(3)} + \cdots + X^{(n)}, \text{ denoted } X^n.$$

$$\begin{aligned} \text{Thus } X^n &= Y + AY + A^2Y + A^3Y + \cdots + A^nY \\ &= [I + A + A^2 + A^3 + \cdots + A^n]\,Y \end{aligned} \tag{2.20}$$

The vector of outputs X^n obtained for a sufficiently large value of n is effectively the same as that obtained by multiplying the vector of final demands by the interdependence coefficients. Hence this "iterative" approach is another way of obtaining the total effect of demand increases upon the outputs of the different sectors. Indeed after five of the above iterations the sum of the success-

TABLE 2.7 Simplified Input–Output Transactions Table, Netherlands, 1956 (unit:

	Outputs → / Inputs ↓		Intermediate Demand				
			Agriculture, Fishing Food	Metals & Construction	Textiles & Apparel	Mining, Chemicals & Utilities	Trade
			(1)	(2)	(3)	(4)	(5)
			I. Intermediate Production and Consumption				
Intermediate Inputs	Agriculture, Fishing, Food	1.	6 419	-	39	175	1
	Metals and Construction	2.	422	3 446	62	399	163
	Textiles and Apparel	3.	25	38	1 177	55	21
	Mining, Chemicals, and Utilities	4.	700	1 438	322	2 210	448
	Trade	5.	404	685	131	274	66
	Services	6.	268	516	141	417	1 241
	Intermediate Inputs (Sub-total)	7.	8 238	6 123	1 872	3 530	1 940
			III. Primary Inputs to Production				
Primary Inputs	Imports	8.	2 547	3 293	1 161	3 016	348
	Depreciation	9.	403	287	95	623	268
	Net Indirect Taxes	10.	638	525	50	492	1 165
	Employees' Income	11.	1 385	3 352	807	1 964	1 239
	Profits	12.	2 865	1 901	432	1 084	2 561
	All Primary Inputs (Sub-total)	13.	7 838	9 358	2 545	7 179	5 581
	Total Input	14.	16 076	15 481	4 417	10 709	7 521

Source: United Nations, Series F, No.14. op. cit., Table 1.1, by courtesy of United

millions of guilders)

		Final Demand							
Services	Total Intermediate Demand	Exports	Household Consumption	Government Consumption	Gross Domestic Fixed Capital Formation	Net Increase in Stocks	Total Final Demand	Total Demand = Total Output
(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
		II. Final Output of Production Sectors						
352	6 986	3 532	5 576	15	-	—33	9 090	16 076
856	5 348	2 747	842	903	5 339	302	10 133	15 481
39	1 355	790	2 146	22	10	94	3 062	4 417
704	5 822	2 630	1 722	341	147	47	4 887	10 709
89	1 649	1 483	3 833	37	509	10	5 872	7 521
1 046	3 629	3 893	3 735	225	248	65	8 166	11 795
3 086	24 789	15 075	17 854	1 543	6 253	485	41 210	65 999
		IV. Primary inputs to Final Demand						
1 614	11 979	368	1 683	319	1 856	238	4 464	16 443
1 128	2 804	-	-	179	-	-	179	2 983
224	3 094	-	-	-	-	-	-	3 094
3 238	11 985	-	-	2 766	10	-	2 776	14 761
2 505	11 348	-	-	106	-	-	106	11 454
8 709	41 210	368	1 683	3 370	1 866	238	7 525	48 735
11 795	65 999	15 443	19 537	4 913	8 119	723	48 735	

Nations, Statistical Office, New York.

TABLE 2.8 Technical Coefficients, Netherlands, 1956, based on Table 2.7

	Inputs ↓	(1)	(2)	(3)	(4)	(5)	(6)
1.	Agriculture, Fishing, Food	0·3993	-	0·0088	0·0163	0·0001	0·0298
2.	Metals and Construction	0·0263	0·2226	0·0140	0·0373	0·0217	0·0726
3.	Textiles and Apparel	0·0016	0·0025	0·2665	0·0051	0·0028	0·0033
4.	Mining, Chemicals and Utilities	0·0435	0·0929	0·0729	0·2064	0·0596	0·0597
5.	Trade	0·0251	0·0442	0·0297	0·0256	0·0088	0·0075
6.	Services	0·0167	0·0333	0·0319	0·0389	0·1650	0·0887
Intermediate Inputs (Sub-total)		0·5124	0·3955	0·4238	0·3296	0·2579	0·2616
All Primary Inputs (Sub-total)		0·4876	0·6045	0·5762	0·6704	0·7421	0·7384
Total Input		1·0000	1·0000	1·0000	1·0000	1·0000	1·0000

TABLE 2.9 Inter dependence Coefficients, Netherlands, 1956

	Inputs ↓	(1)	(2)	(3)	(4)	(5)	(6)
1.	Agriculture, Fishing, Food	1·6700	0·0078	0·0271	0·0382	0·0124	0·0580
2.	Metals and Construction	0·0673	1·3025	0·0396	0·0699	0·0513	0·1111
3.	Textiles and Apparel	0·0049	0·0061	1·3649	0·0097	0·0056	0·0063
4.	Mining, Chemicals, Utilities	0·1071	0·1632	0·1399	1·2789	0·0977	0·1016
5.	Trade	0·0486	0·0632	0·0474	0·0379	1·0156	0·0177
6.	Services	0·0466	0·0664	0·0643	0·0651	0·1904	1·1102

ive rounds is nearly identical with the "inverse" solution. Accordingly we have here an alternative method of inverting an $(I-A)$ matrix, i.e. where n is large

$$X^n = [I+A+A^2+\cdots+A^n]\,Y \doteq (I-A)^{-1}Y$$

thus $$(I-A)^{-1} \doteq [I+A+A^2+\cdots+A^n] \qquad (2.21)$$

where $\doteq$ means " approximately equal to" and where I is a unit matrix and A is a matrix of technical coefficients. This method of matrix inversion is known as the "expansion of powers method" but it is only suitable for matrices whose elements from a convergent* series when expanded in this way.

APPENDIX TO CHAPTER 2

The three tables 2.7 to 2.9 are given as a further example of the successive stages of input–output analysis. Table 2.7 shows the Transactions for the Netherlands (1956) in a system of six productive sectors. Indirect Taxes less Subsidies are denoted as "Net Indirect Taxes", while Depreciation is shown separately from Profits, among the Primary Input rows. There are five columns shown for the components of Final Demand. Table 2.8 shows the Technical Coefficients with all Primary Inputs combined in a single row, while Table 2.9 sets out the Interdependence Coefficients.

EXERCISES

Exercise 2.1 The following is a hypothetical input–output table

TABLE A £m

Inputs	Inter-Industry			Final Demand	Total Output
	(1)	(2)	(3)		
Agric. 1	-	20	15	40	75
Industry 2	4	10	-	60	74
Services 3	8	-	12	50	70
Primary Inputs					
Imports	20	10	3	-	33
Subsidies	−6	−4	-	-	−10
Indirect Taxes	2	8	5	-	15
Income arising	47	30	35	-	112
Total Inputs	75	74	70	150	369

* Convergence is assured if the sum of the elements in each column is equal to or less than unity and at least one column total is less than unity (see Chenery and Clark, ref. 1 of Introduction, p.52).

(a) Calculate technical and interdependence coefficients for this model using the adjoint matrix method* and verify that the latter coefficients can be obtained approximately from the technical coefficients by the "expansion of powers" method (see formula (2.21)).
(b) Using the matrices of interdependence and technical coefficients, derive the transactions table appropriate to the following levels of final demand:
Sector 1 – £45m, Sector 2 – £70m, Sector 3 – £60m.

Exercise 2.2 The following table gives the $(I-A)^{-1}$ results for the United Kingdom 1963 table of technical coefficients (2.4).

TABLE B Interdependence Coefficients for United Kingdom 1963

Sector ↓	Agriculture	Industry	Services
Agriculture	1·1766	0·0280	0·0042
Industry	0·5988	1·6265	0·1841
Services	0·2195	0·1754	1·1142

(a) Pre-multiply the elements of Total Final Demand appearing in Table 2.3 by the elements of Table B and compare the results with the values of output shown in the first three entries of Column (9) of Table 2.3. The two sets of figures should be identical, except for rounding errors. Multiplications should be carried out as in (2.14) above.
(b) Take each of the columns (5), (6) and (7) of Table 2.3 and pre-multiply each in turn by Table B matrix. This gives the direct plus indirect output of each productive sector required by Consumption, Capital Formation and Exports. Check that the column vector, given by the sum of the three column vectors obtained from the pre-multiplications, agrees with the first three elements of column (9) of Table 2.3, subject to rounding errors.

Exercise 2.3

(a) Multiply each of the first six elements in Column 13 of Table 2.7 by 2 to obtain a new vector of total final demand twice as great as the original.
(b) Pre-multiply each element of this new vector by the inverse coefficient matrix shown in Table 2.9 to obtain a new output vector.
(c) Verify that each element of this new output vector is twice as great as the corresponding element in Column 14 of Table 2.7.
(d) Apply the technical coefficients given in Row 2 of Table 2.8 to the new output vector. Sum the results and verify that this sum is twice the level of total intermediate given in Row 2 of Table 2.7 (i.e. 5348 x 2 = 10696).

* See mathematical Appendix.

3 MULTIPLIER ANALYSIS AND PRICE EFFECTS IN THE FRAMEWORK OF ECONOMIC PLANNING

The natural setting for multiplier analysis, more recently called *Impact Analysis,* is economic planning. In the present chapter there are three main sections. The first of these considers an elementary approach to planning, by using I—O to produce sector outputs and primary inputs consistent with specified final demands. Within the framework of this same approach to planning, final demand multipliers, which generate income or employment, are discussed.

The second section denoted *Impact Analysis* describes relatively specialized development in multiplier analysis, whereby household income and household expenditure are included as part of the "augmented" inter-industry matrix.

The third section describes *Price Effects* arising from changes in primary input coefficients and working their way consistently through the economic system. Rigidity of structure and a complete lack of demand adjustment to price changes are postulated for the inter-industry matrix. The price calculations, as shown, are based on the latter matrix, without inclusion of a household row and column. Some aggregate price effects are illustrated by numerical examples.

AN ELEMENTARY APPROACH TO PLANNING

Using the System to Plan an Economy

The input—output system may be used in a number of different ways for the purpose of economic planning. The method to be described here is known as the consistency approach whereby a target final demand vector is pre-multiplied by the matrix of interdependence coefficients to determine the production requirements. A more advanced method which should be mentioned is the mathematical programming approach used by Geary[1] in preparing his decision model for Ireland based on the 1960 Input—Output table. A somewhat similar type of linear programming exercise is shown in Chapter 9 based on the UK structure given in Tables 2.3 and 2.4. Chapter 9 also includes a brief introduction to the concepts involved, but for a more advanced treatment readers are referred to Chapters 4 and 11 of Chenery and Clark[2] where the relevant techniques and their applications are explained in some detail.

Consistency Method

Where this approach to planning is adopted a vector of final demands for the different products in some future year must be specified and this vector is then multiplied by the matrix of interdependence coefficients. We describe later some of the problems involved in this method of planning but before doing so we illustrate below how the calculations are carried out using our highly aggregated simple Irish model. For this purpose let us assume that final demand (i.e. household and government consumption, capital formation, and exports) for agricultural produce increases by say £24·665 million from £115·335 million (figure given in Table 2.1) to £140 million, that final demand for industrial products increases from £386·917 million to £447 million and that final demand for services increases from £238·562 million to £278 million. By substituting the new values into equation system (2.8) of Chapter 2 the resulting outputs are £239 million for agriculture, £623 million for industry and £351 million for services.

The substitutions are made as follows:

$$\begin{aligned} X_1 &= 1{\cdot}0394(140) + 0{\cdot}1945(447) + 0{\cdot}0218(278) = 238{\cdot}5 \\ X_2 &= 0{\cdot}1833(140) + 1{\cdot}2652(447) + 0{\cdot}1150(278) = 623{\cdot}2 \qquad (3.1) \\ X_3 &= 0{\cdot}0729(140) + 0{\cdot}0925(447) + 1{\cdot}0778(278) = 351{\cdot}2 \end{aligned}$$

By applying the technical coefficients in Table 2.2 to these new output levels the internal flows for the planned system are obtained. These are shown in Table 3.1. Rounding errors prevent exact agreement between entries and total, for the second and third columns.

As can be seen from Table 3.1 the inter-industry flows are shown in detail, but aggregated figures only are given for the primary inputs (and of course for the final demands which were specified in advance). Accordingly estimates of the composition of the latter items must be made. In an actual case the model would contain a large number of sectors both industrial and agricultural and it would be necessary to decompose each sector separately.

There are various ways of decomposing the primary inputs, depending on the sector involved. One way would be to assume that the different inputs in any given sector in the planned year bore the same relationship to output as they did in the base year. This might be a reasonable assumption for some sectors but not for others. For example, it might not be realistic to say that the level of income arising in the planned year accounted for the same proportion of output in all sectors as it did in the base year. Most planners would probably prefer to leave income as a residual, and predict the level of the other primary inputs on the basis of information from outside as well as from within the system.

Various methods are used in specifying final demands, from simple extrapolation of past trends to more elaborate methods using econometric techniques.

TABLE 3.1 Internal Flows for Planned Level of Final Demand (£ million)

Inputs ↓	Agriculture (1)	Industry (2)	Services (3)	Final Demand	Output
Agriculture	2·6	94·6	1·3	140·0	238·5
Industry	33·0	113·5	29·7	447·0	623·2
Services	13·1	37·3	22·7	278·0	351·2
All Primary Inputs	189·8	377·7	297·5		
Total Inputs	238·5	623·2	351·2		

In the latter case final demands for certain aggregated groups of commodities are considered as being dependent on a number of other variables such as population growth, trends in consumption patterns, exports, trade agreements, etc. The variables are expressed in the form of simultaneous equations and the system solved to obtain the projected aggregated variables which are used to obtain disaggregated final-demand items. In the specification of the latter, care must be taken to preserve consistency between sectors and this is especially true in the specification of final demands for individual agricultural commodities because of the complementary nature of certain enterprises. Cattle and dairying is a case in point. If we were to specify an increase in the demand for cattle exports with no increase in the final demand for dairy products most of the increased exports would be shown as coming from increased imports. This would be realistic enough for a very small increase in cattle exports but if a large increase were planned the result would not be realistic. Hence if a sizeable increase in final demand for cattle is proposed, a consistent increase in final demand for dairy produce must be planned also. In a similar way, if final demand for dairy produce is specified to increase by a certain amount, a corresponding increase must also be planned for the final demand for either live cattle or beef, and so on for other enterprises. Consistent estimates of final demand can be specified in various ways depending on the sectors involved. For some sectors it may be reasonable to specify that final demands in the planned year are in the same proportions as they were in the base year.

For other sectors more elaborate figuring may have to be carried out, using data from outside the system. In cases however where the products of several sectors are complementary to one another, e.g. cattle, dairying, pigs, milk-processing, feed grains etc., the specifications become more complicated and it may be necessary to experiment with different levels before arriving at consistent final demands.

In planning, therefore, the administrator must set realistic and consistent targets for final demand. He can then examine with the aid of input–output the effect of these targets on the economy as a whole. If it is found from this

examination that the set targets give unrealistic internal flows, revised projections will have to be made until a realistic system is obtained. Usually the goals set cannot be achieved without State direction. Therefore, as soon as a consistent plan has been drafted, policy must be framed to achieve the overall targets.

One of the biggest problems associated with plans based on an input–output model concerns the stability of the technical coefficients over time. If the model is used without adjustment the planner is assuming that the inter-industry relationships within the economy will be the same in the year for which the plans are made as they were in the base year. This is not usually a valid assumption. It is well known that the structural coefficients vary with time because of changing technology and scale of enterprise. These changes bring about substitution effects which are difficult to anticipate and which can cause considerable change in production and consumption patterns. These hazards are of course encountered no matter what method of planning is adopted but they are especially critical when the input–output method is used because of the inflexible nature of the system.

It is possible to alter the coefficients by making estimates of coefficient changes over time and systematic methods for making such alterations are available. The best known of these is probably the RAS method, devised by Stone and his associates at Cambridge[3], which is explained in some detail in Chapter 6.

IMPACT ANALYSIS

In the previous chapter it was shown that every £1 final demand for the products of a sector generates indirect as well as direct income effects on the economy as a whole. The relationship between the initial spending and the total effects generated by the spending is known as the *multiplier* effect of the sector, or more generally as the *impact* of the sector on the economy as a whole. For this reason the study of multipliers has come to be called *impact analysis.* Before taking up this study however, it is useful to refer briefly to some theoretical ideas on which the subject is based.

Keynesian Concepts

Though the original multiplier idea can be traced back to Kahn's work in 1931[4] the modern concept of an income multiplier is usually associated with J.M. Keynes and might be described as follows[5]. A unit increment of "autonomous" investment causes an initial increase in income which generates successive rounds of consumer spending and incomes, each round producing numerically smaller increments until the process has fully worked itself out, i.e. has reached equilibrium. The fully worked-out response to the stimulus produces (a) savings equal to the initial unit increment of investment, and (b) consumer spending (household consumption) considerably larger than the initial unit increment of invest-

ment. The household consumption is a multiple of the unit increment of investment, the multiplier being given by $1/(1-c)$, where c is the marginal propensity to consume. Other assumed autonomous expenditures like government spending and exports have a similar effect.

The above explanation gives an over-simplified picture of reality and an exaggerated estimate of the size of the multiplier, because both government taxation and purchase of imports reduce the size of the multiplier, so that the latter is considerably smaller than $1/(1-c)$, in normal present-day economic conditions. What has been made clear, however, is the principle of household consumption interacting with household income through successive rounds, so as to produce an increment of household consumption larger than the initial unit of autonomous investment, at the equilibrium level.

Partial Multipliers

The simplest type of income multiplier which can be calculated is what might be described as a *partial multiplier* so as to distinguish it from more complete multipliers which will be described later. This type of multiplier for a particular sector is calculated from the input–output system by multiplying the row of technical coefficients of income arising in each sector by the column of interdependence coefficients of the sector concerned. It will be noted that all of these partial multipliers are less than unity and indeed by definition they cannot exceed this amount. The extent to which they are less than 1.0 depends on the import content, tax rate, and retained profits of the economy concerned. In a closed economy with no taxes the partial multipliers would be 1.0, i.e. income and expenditure would be identical concepts. In an open taxed economy the magnitude of the multipliers will depend on the tax and import rates and on other leakages. The greater these are the smaller the multipliers.

To understand how this multiplier is calculated the reader is referred to Table 3.2 where are given the interdependence coefficients for our simple three-sector Irish model together with the technical coefficients for income arising, i.e. wages, salaries and profits, in the different sectors.

We have shown in Chapter 2 that, if final demand for the products of any sector is increased by one unit with no change in final demand for the products of the other sectors, the resulting outputs of the different sectors are given by the corresponding column of interdependence coefficients. Thus it was shown that, if the final demand for agricultural goods was increased by one unit, the output of agriculture would be increased by 1·0394, that of industry by 0·1833, and that of services by 0·0729. If we look now at the technical coefficients of wages, salaries and profits in Table 3.2, we see that the coefficient for agriculture is 0·6668. Hence an increase of 1·0394 units in agricultural output will increase the income arising in that sector by 0·6931 units (i.e. 1·0394 × 0·6668). Similarly an increase of 0·1833 in the output of industry will increase industrial income by 0·0512 (i.e. 0·1833 × 0·2795) while an increase of 0·0729 in the

TABLE 3.2 Interdependence Coefficients for Intermediate Sectors and Technical Coefficients for Income Arising

Sector	Agriculture	Industry	Services
	Interdependence Coefficients		
Agriculture	1·0394	0·1945	0·0218
Industry	0·1833	1·2652	0·1150
Services	0·0729	0·0925	1·0778
	Technical Coefficients		
Wages, Salaries, Profits (Income arising)	0·6668	0·2795	0·7619

output of services will increase the income of that sector by 0·0555 (i.e. 0·0729 x 0·7619). The benefit to the whole economy of a unit increase in final demand for the products of agriculture is therefore an increase of 0·7998 units in the income of the nation (i.e. 0·6931 + 0·0512 + 0·0555).

The various steps shown above may appear rather involved but they have been carried out in order to show the logic of the procedure. In practice, however, the calculations are very simple and can be done quickly in a systematic manner. A study of the above figures shows that since final demand for any sector may be increased by one unit with zero increases in the other sectors the exercise resolves itself simply into multiplying each element in the column of interdependence coefficients of the sector by the corresponding element in the transposed row* of technical coefficients for income arising, and summing the results. The systematic calculations for the three sectors are shown in Table 3.3.

TABLE 3.3 Calculation of Partial Income Multipliers

Sector	Technical Coefficient of Income Arising	Interdependence Coefficients			Income Arising		
	(1)	Ag. (2)	Ind. (3)	Ser. (4)	Ag. (1)x(2)	Ind. (1)x(3)	Ser. (1)x(4)
Agriculture	0·6668	1·0394	0·1945	0·0218	0·6931	0·1298	0·0145
Industry	0·2795	0·1833	1·2652	0·1150	0·0512	0·3536	0·0321
Services	0·7619	0·0729	0·0925	1·0778	0·0555	0·0705	0·8212
Total Income Arising (Partial Income Multipliers)					0·7998	0·5539	0·8678

* As explained in the Mathematical Appendix, a transposed row is a row written as a column. Column (1) of Table 3.3 is the transpose of the row of technical coefficients in Table 3.2.

TABLE 3.4 Input–Output Table for Ireland 1960, having Households included in

Inputs ↓ / Outputs →	Inter-Industry			
	Agriculture (1)	Industry (2)	Services (3)	Households (4)
1. Agriculture	2·180	81·687	1·143	62·111
2. Industry	27·709	98·036	25·457	207·086
3. Services	11·020	32·242	19·487	139·195
4. Households	133·600	134·082	204·657	-
5. Total Inter-Industry	174·509	346·047	250·744	408·392
6. Imports	15·294	119·842	7·855	62·295
7. Indirect taxes	11·559	49·257	9·200	31·884
8. Subsidies	−7·317	−5·848	−6·701	-
9. Residue of Wages, Salaries, Profits etc.	-	16·321	24·913	-
10. Depreciation	6·300	12·500	15·300	-
11. Total Primary Inputs	25·836	192·072	50·567	94·179
12. Input = Output	200·345	538·119	301·311	502·571
13. Persons employed (thousand)	242*	248	417	-

* Estimated labour units required in agriculture instead of amount available.

Inter-Industry Transactions. £ million.

Total Inter-Industry	Final Demand				Output
	Government Consumption	Capital Formation	Exports	Total Final Demand	
(5)	(6)	(7)	(8)	(9)	(10)
147·121	0·803	2·671	49·750	53·224	200·345
358·288	14·821	61·732	103·278	179·831	538·119
201·944	50·849	6·428	42·090	99·367	301·311
472·339	-	-	30·232	30·232	502·571
1 179·692	66·473	70·831	225·350	362·654	1 542·346
205·286	1·764	25·983	3·345	31·092	236·378
101·900	-	0·886	3·175	4·061	105·961
−19·866	-	-	-	-	−19·866
41·234	-	-	3·680	3·680	44·914
34·100	2·500	−1·100	-	1·400	35·500
362·654	4·264	25·769	19·200	40·233	402·887
1 542·346	70·737	96·600	235·550	402·887	1 945·233
907	-	-	-	-	907

These results show that the partial income multiplier of a unit of final demand for agricultural produce is 0·7998, that of a unit of industrial produce is 0·5539, while that of a unit of services is 0·8678.

Partial Multipliers for Other Primary Inputs

Partial multipliers for any other primary input row may be calculated in exactly the same way, i.e. by multiplying the interdependence coefficients by the technical coefficients of the row concerned. Import multipliers of this kind are of special interest as they show the import requirements of a unit of final demand for the produce of each sector and how the balance of trade is affected by specific increases in the final demands for the products of the different sectors. The partial import multipliers for the three sectors in our simplified model in 1960 were: agriculture 0·1221, industry 0·2990, and services 0·0554. In a similar manner multipliers for subsidies, indirect taxes, and depreciation can be calculated from the model.

Partial employment multipliers can also be calculated, but in order to do this, employment coefficients have to be calculated. These coefficients are usually expressed as persons employed per £1 total output, but several rows of coefficients can be calculated, each row measuring a specified category of labour, e.g. managerial and clerical, skilled manual, unskilled, etc. For each category of labour a row of partial labour multipliers can be derived corresponding to the partial income multipliers forming the last row of Table 3.3. Capital multipliers could be calculated in the same way as labour multipliers but since capital is a stock rather than a flow resource the meaning of a capital multiplier requires very careful interpretation.

Complete Multipliers

As stated above the income multipliers calculated in Table 3.2 are all less than unity, indicating that £1 final demand for the products of any of the three sectors generates less than £1 income in the economy as a whole, even when all the different round effects have worked themselves out. We have, therefore, referred to them as partial multipliers because most of the Keynesian multipliers referred to in the literature are much greater than unity.

Indeed it is impossible to derive proper multipliers of the Keynesian type from the type of input–output tables which have so far been presented in this book. In all these tables, household income is considered as being outside the inter-industry matrix, and the expenditure of this income is therefore treated as a leakage from the system rather than as a generator of further economic activity within it. Hence if proper Keynesian-type multipliers for different sectors

are to be derived from an input–output table, households must be included in the inter-industry section. When this is done the household income is treated as being spent within the system and as generating further economic activity.

The inclusion of household income in the inter-industry matrix does however present some slight technical problems because, in addition to the household column which is already available in the final-demand quadrant, a corresponding row must be included so as to keep the matrix square. No such row is directly available in a conventional input–output table, and one must therefore be derived from the wages, salaries, and profits entries, since the latter items are the main sources of household income.

Table 3.4 is the highly aggregated Irish Transactions Table (Table 2.1) re-arranged to include a row and column for Households as a fourth inter-industry sector. Column (4), for Households, is identical with that for Household Consumption in Table 2.1, having £502·571 million as total input. Row (4), denoted "Households" also, having a row total of the same aggregate value, £502·571 million, has been obtained by taking the entries of Row (8) of Table 2.1 (namely Wages, Salaries, Profits etc.) either in whole or in part. The residual parts of Row (8) of Table 2.1 appear as Row (9) of Table 3.4 and are denoted "Residue of Wages, Salaries, Profits etc.". This row includes mainly undistributed profits of companies, direct taxes, and savings other than those included in the depreciation row. It should be emphasized that this model is purely for demonstration purposes; a more precise model would need to have rows for government income and savings. By specifying carefully what is included or excluded from the household column and row we can generate multipliers relating to income at factor cost or at market prices. In this example the multiplier relates to disposable income which is at market prices. Apart from Rows (4) and (9), all the other rows and columns of Table 3.4 are identical with the correspondingly named rows and columns of Table 2.1, with the exception of Row (13) which is a new row showing the numbers employed in the different sectors in 1960. As there was considerable under-employment of labour in agriculture in Ireland in that year the *estimated* farm labour requirement[6] rather than the labour available has been used in deriving this row. A multiplier based on the amount of labour available rather than on the amount required would overstate considerably the employment potential of new agricultural enterprises. Table 3.5 has the technical or direct input coefficients derived from transactions of Table 3.4. Rows (4) and (11) are treated like any other input rows and Column (4) just like Columns (1) to (3).

Table 3.6 has the next stage of the standard input–output analysis of an inter-industry matrix of dimension 4. The top part shows the (I – O) inverse denoted "Interdependence Coefficients" and the bottom part has the multipliers for the different primary inputs and persons employed. Each row of this part of Table 3.6 has been calculated in the same way as the row of partial income multipliers in Table 3.3.

TABLE 3.5 Technical and Employment Coefficients for Irish 1960 I – O Table having Households included in Inter-Industry Transactions

Inputs ↓	Agriculture (1)	Industry (2)	Services (3)	Household Consumption (4)
1. Agriculture	0·010881	0·151801	0·003793	0·123587
2. Industry	0·138306	0·182183	0·084487	0·412053
3. Services	0·055005	0·059916	0·064674	0·276966
4. Household income	0·666850	0·249168	0·679222	0·0
Primary				
5. Imports	0·076338	0·222705	0·026069	0·123953
6. Indirect taxes	0·057695	0·091536	0·030533	0·063442
7. Subsidies	−0·036522	−0·010867	−0·022239	0·0
8. Residue of wages etc.	0·0	0·030330	0·082682	0·0
9. Depreciation	0·031446	0·023229	0·050778	0·0
10. Total Input	1· -	1· -	1· -	1· -
11. Employment coefficients (Number of persons per £1 000 Total Input)	1·207916	0·460865	1·383952	0·0

Because households are now interacting with other sectors these primary input multipliers are of the Keynesian type and show the total effect on imports, indirect taxes, subsidies, depreciation, labour etc. of £1 final demand (as defined) for the products of the different sectors. For example the import multipliers indicate that every £1 spent on either agricultural exports, capital formation or government consumption for sector (1) generates £0·58 of demand for imports and £0·31 of indirect taxes. Every £1 spent on industrial goods generates £0·59 of import demand and £0·28 indirect taxes and so on for the other primary inputs. The employment multipliers show that every £1 000 of final demand for agricultural products generates employment in the whole economy for 3·1 labour units. Every £1 000 industrial final demand generates employment for 2·0 units of labour. The labour multiplier for services is 3·2 and that for household consumption is 2·1.

From a cursory examination of Table 3.6 it might appear that we cannot derive multipliers for income arising in the different sectors since we have now brought almost all of this item into the inter-industry quadrant of the matrix. This however is not so, because the interdependence coefficients of the Household Income row are now the household income multipliers, and when these are added to the multipliers given in Table 3.6 for Residue of Wages etc. we

TABLE 3.6 Interdependence Coefficients with Primary Input and Employment Multipliers derived from Table 3.5

Inputs ↓	Agriculture (1)	Industry (2)	Services (3)	Household Consumption (4)
Interdependence Coefficients				
1. Agriculture	1·392944	0·422261	0·369468	0·448473
2. Industry	1·131756	1·876173	1·047777	1·203152
3. Services	0·642130	0·459185	1·637633	0·722136
4. Household Income	1·647030	1·060955	1·619769	2·089340
Primary Input Multipliers per Unit Final Demand				
Imports	0·579277	0·593547	0·505017	0·579989
Indirect Taxes	0·308059	0·277428	0·269989	0·290606
Subsidies	−0·077453	−0·046023	−0·061301	−0·045514
Residue of Wages etc.	0·087418	0·094870	0·167182	0·096199
Depreciation	0·102698	0·080177	0·119113	0·078719
Total Primary	0·999999	0·999999	1·000000	0·999999
Number of persons required per £1 000 final demand	3·092823	2·010208	3·195575	2·095611

obtain the total multipliers for income arising. The total income multipliers are:

$$\begin{array}{lrcl} \text{Agriculture etc.} & 1{\cdot}647030 + 0{\cdot}087418 & = & 1{\cdot}734448 \\ \text{Industry} & 1{\cdot}060955 + 0{\cdot}094870 & = & 1{\cdot}155825 \\ \text{Services} & 1{\cdot}619769 + 0{\cdot}167182 & = & 1{\cdot}786951 \\ \text{Household Consumption} & 2{\cdot}089340 + 0{\cdot}096199 & = & 2{\cdot}185539 \end{array} \qquad (3.2)$$

To verify that these are the true multipliers for income arising we multiply them by the entries in the first four rows of Column 9 of Table 3.4 (i.e. the column of total final demand) to obtain the total income generated by the four sectors in question. The calculations are as follows:

$$(1{\cdot}734448 \times 53{\cdot}224) + (1{\cdot}155825 \times 179{\cdot}831) + (1{\cdot}786951 \times 99{\cdot}367) + (2{\cdot}185539 \times 30{\cdot}232) = 543{\cdot}804$$

Except for a slight rounding error this figure is equal to 543·805 which is obtained by adding 502·571 (Row 4 Column 10 of Table 3.4) to 41·234 (Row 9

Column 5 of Table 3.4).

The multipliers at (3.2) above show that every £1 of final demand for agricultural produce (i.e. exports, government consumption, and capital formation) generates total income of £1·73 throughout the whole economy. Similarly the multiplier for industry is 1·16, that for services is 1·79 and that for household consumption itself is 2·19. The latter figure means that every £1 of direct final demand for the household row (say direct transfers of cash by the government to households) is more than doubled when it has fully worked out its impact through the entire economy.

These multipliers are smaller than might be expected from a reading of the economic literature on the subject. In most text books we find average impact effects of well over 2·0 for whole economies. The low Irish figures are due to the open nature of the economy, imports amounting to about 40 per cent of GNP. This level of imports causes a severe leakage from the system, resulting in relatively low multipliers for many sectors, particularly for some industrial enterprises which depend heavily on imported raw materials.

In interpreting the multipliers shown in Table 3.6 it is important to note that we cannot distinguish (if any difference exists) between the effects of expenditure on any of the three items now remaining in final demand, namely Government Consumption, Capital Formation, Exports. In other words £1 expenditure on industrial exports is assumed to have the same multiplier effect as £1 capital formation or £1 spent by the government on industrial items. In the long run these types of expenditure may each have different effects but in the short run (to which this type of model relates) the source of the demand expansion is irrelevant. In other words, in a single accounting period £1 capital formation has exactly the same effect on income arising in an economy as £1 exports.

Other Multipliers

A few other types of sectoral multipliers have been described in the input–output literature[7],[8] and, though we have some doubts about their validity, we will describe one of them here as it seems to have some relevance for certain types of analysis. The multiplier in question was first defined and applied by Moore[9] and by Moore and Peterson[10] in 1955 and hence we refer to it here as a Moore Type-2 multiplier.* The derivation of this multiplier is shown in Table 3.7.

As can be seen from Table 3.7 the Moore Type-2 multipliers are obtained by dividing the interdependence coefficients from the first three columns of Row 4 of Table 3.6 by the technical coefficients for Household Income from the first three columns of Table 3.5. Because the divisors are less than unity

* There is also a Moore Type-1 multiplier which we do not define but which is described in references [9] and [10].

TABLE 3.7 Derivation of Moore Type-2 Multipliers for Irish 1960 data

Sector	Agriculture	Industry	Services
(1) Technical coefficient for household income from Table 3.5	0·6668	0·2492	0·6792
(2) Keynesian-type multipliers (Row 4 Table 3.6)	1·6470	1·0610	1·6198
(3) Moore Type-2 multipliers (2)/(1)	2·4700	4·2576	2·3849

the Moore multipliers are all greater than the corresponding Keynesian-type ones, that for industry at 4·258 being exceptionally high for a sector which has a very large import content. Indeed the levels of these Moore-type multipliers are roughly in inverse proportion to the relative magnitudes of the income arising in different sectors and very roughly in direct proportion to the levels of imports of the sectors.

From the manner of their construction, multipliers of the Moore Type 2 are not suitable for application to final demands. They are suitable only for applying to income arising in sectors. Thus, if we know that the magnitude of the Moore Type-2 multiplier for a particular industry is 3.0, we can say that every £1 income earned in that industry generates £3 household income in the economy as a whole. Hence a Moore Type 2 multiplier for a sector of an economy might be useful for application, within that sector, to an industry for which information was lacking regarding disposal of output but was available in regard to income arising.

Comments on the Application of Sectoral Income and Employment Multipliers

Income and employment multipliers for sectors of an economy and for individual industries have many practical applications and have been widely used in various studies throughout the world. It would appear, however, that they are liable to be misused, particularly by interested groups who wish to show larger incomes from industries than are warranted by the facts. Because of possible misuse, multiplier analysis is now regarded somewhat suspiciously by economists and we must therefore consider carefully the conditions under which the use of multipliers is justified.

It might first be said that multipliers have their widest application in marginal-type analysis, i.e. in examining the effect, on an economy or a region, of starting a new industry or of discontinuing an existing one. Their valid application in other situations is limited, although they could be used to quantify the superiority of one existing industry over another with a view to changing the industrial mix of an economy. Thus engineering and textile industries could be compared

by the use of income and employment multipliers and, on the basis of the results, an industrial strategy could be planned, taking into account the relative scarcities or otherwise of capital and labour and the availability of raw materials. An analysis such as this would enable a more realistic appraisal of trade-offs between capital-intensive and labour-intensive industries.

If multipliers are to be validly used to show the effect, on an economy or region, of introducing a new industry then two assumptions must be made:
(1) There are unemployed or under-utilized resources in the region, otherwise the introduction of the new industry will do no more than shift resources from one sector to another, from say agriculture to industry and so on. A shift of the kind contemplated might of course be most useful, but in order to test its desirability, comparative analysis of all the existing industries should be made.
(2) The operation of the new industry must be associated with an increase in final demand in the economy as defined, i.e. government expenditure, capital formation or net exports (exports less imports). If it is not, the new industry does no more than displace an existing one and even if it uses hitherto unemployed resources in one area it will cause unemployment of resources in another. The multiplier for a new industry should therefore be applied only to the portion of its produce which increases final demand either directly through exports, government consumption or capital formation, or indirectly through substitution for imported goods.

It should be kept in mind of course, that the application of a multiplier to the products of a new industry shows the effect only of the production of the goods, and takes no account of the effect on the economy of building the factory or plant. The total effect can however be taken into account by a two-stage computation. In the first stage a multiplier for new construction (if such is available) can be applied to the investment used in building the factory and the result can be taken as the effect of this investment on incomes and employment during the construction period. The second stage consists of applying the industry multiplier to the exports (and possibly to some of the home consumption) of the new product in order to assess the long-term effects of the industry itself. The latter will come into effect only when the construction of the factory is completed and the multiplier effects of the latter expenditure are likely to have worked themselves out.

One of the biggest practical difficulties connected with multiplier analysis is the availability of suitable multipliers for a given situation. If the industry under review is one which produces finished goods for export there is usually no great problem involved, but if it is an industry which will produce goods heretofore imported as raw materials for an existing industry, then the inter-industry structure may be seriously affected by the proposed industry, and the existing multiplier or multipliers may be inapplicable. In such circumstances the only thing which can be done is to adjust the existing technical coefficients

to take account of the new hypothetical situation and from these calculate new multipliers.

The above discussion on the application of multipliers is far from exhaustive but it should be sufficient to alert workers in the field to some of the pitfalls involved in this type of analysis. The main points emerging from the discussion are that multiplier analysis can be a useful exercise if performed sensibly and without bias. Multipliers, however, cannot be applied indiscriminately and without very careful consideration of the existing economic structure. Furthermore, suitable multipliers for a given situation may not be available and the question of producing new ones may have to be considered. It might be well to point out also that in all of this discussion prices are assumed fixed. In other words real and nominal incomes are the same. The effect of price changes is discussed in the next section.

PRICE EFFECTS

A variation of the income multiplier calculation is the estimation of the output price changes which are necessary to increase sectoral incomes by a given amount with no change in the quantities of the various commodities produced or in the manner of disposal.* The method of doing this has been explained elsewhere by Geary and Pratschke [11]. The formula (with a very slight modification) given by these authors, is as follows:

$$P = [(I-A)^{-1}]'(\pi b) \qquad (3.3)$$

where: P is the column vector of output price changes, (πb) is a transposed row vector of primary inputs (i.e. written as a column vector) after the postulated changes have been applied; b being the vector of technical coefficients of some primary input (income arising, imports etc.) in the base year, and π being the percentage changes postulated in the primary inputs (different changes may be postulated for different inputs). $[(I-A)^{-1}]'$ is the transposed inverse matrix, i.e. the inverse matrix with its rows written as columns. An alternative formulation which may be more convenient for application is

$$P' = (\pi b)'(I-A)^{-1} \qquad (3.4)$$

To show how equation (3.3) is derived we take as an example a two-sector input–output model and write out the transactions table as shown in Table 3.8.

The problem is to determine for Sectors (1) and (2) the price changes which are required to give increases of some specified amounts in the levels of primary inputs (including income arising).

* It should be noted that in a closed economy where there are no exports or imports, price increases on the home market can bring about no *real* gain in overall national income, though of course differential price changes will redistribute income between one sector and another.

TABLE 3.8

	Intermediate		Final	Output
	(1)	(2)		
Sector (1) Sector (2)	x_{11} x_{21}	x_{12} x_{22}	Y_1 Y_2	X_1 X_2
Primary Inputs	B_1	B_2		
Total Inputs	X_1	X_2		

TABLE 3.9

	Intermediate		Final	Output
	(1)	(2)		
Sector (1) Sector (2)	P_1x_{11} P_2x_{21}	P_1x_{12} P_2x_{22}	P_1Y_1 P_2Y_2	P_1X_1 P_2X_2
Primary Inputs	$\pi_1 B_1$	$\pi_2 B_2$		
Total inputs = outputs	P_1X_1	P_2X_2		

Let the percentage increase in the primary inputs to Sectors (1) and (2) be specified as π_1 and π_2 respectively and let the required price increases for Sectors (1) and (2) be P_1 and P_2 respectively. The position after the price and income changes have come about is shown in Table 3.9, which is derived by multiplying the entries in Table 3.8 by the relevant price and income changes.

If we were to divide the figures in the first column of Table 3.8 above by X_1 and those in the second column of the table by X_2 we would derive the usual set of technical coefficients, i.e. the a_{ij}'s. Suppose instead we divide the figures in the first and second columns of Table 3.9 by X_1 and X_2 respectively we obtain technical coefficients at base-year prices multiplied by the percentage price and income increases as follows:

	(1)	(2)	
Sector (1) Sector (2)	P_1a_{11} P_2a_{21}	P_1a_{12} P_2a_{22}	(3.5)
Primary Inputs	$\pi_1 b_1$	$\pi_2 b_2$	
Total	P_1	P_2	

where $a_{ij} = x_{ij}/X_j$ and $b_j = B_j/X_j$.

This system can be written in the form of equations as follows:

$$\begin{aligned} P_1 a_{11} + P_2 a_{21} + \pi_1 b_1 &= P_1 \\ P_1 a_{12} + P_2 a_{22} + \pi_2 b_2 &= P_2 \end{aligned} \tag{3.6}$$

From this we obtain

$$\begin{aligned} P_1(1 - a_{11}) - P_2 a_{21} &= \pi_1 b_1 \\ -P_1 a_{12} + P_2(1 - a_{22}) &= \pi_2 b_2 \end{aligned} \tag{3.7}$$

which can be written in matrix form as

$$\begin{bmatrix} (1 - a_{11}) & -a_{21} \\ -a_{12} & (1 - a_{22}) \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} \pi_1 b_1 \\ \pi_2 b_2 \end{bmatrix} \tag{3.8}$$

As in the various input–output equations throughout the text the matrix on the left-hand side of (3.8) is an $(\mathrm{I} - A)$ matrix but in this case it is a transposed $(\mathrm{I} - A)$ matrix, i.e. one in which the rows and columns are interchanged, e.g.

$$\begin{bmatrix} (1 - a_{11}) & -a_{12} \\ -a_{21} & (1 - a_{22}) \end{bmatrix}' = \begin{bmatrix} (1 - a_{11}) & -a_{21} \\ -a_{12} & (1 - a_{22}) \end{bmatrix}$$

If we denote the vector on the right of equation (3.8) by (πb) and if we write the matrix on the left-hand side as $(\mathrm{I} - A)'$, this equation can be written in matrix form as

$$(I - A)'P = (\pi b)$$

$$P = [(I - A)']^{-1}(\pi b) \tag{3.9}$$

The Mathematical Appendix shows that the inverse of a transposed matrix is the same as the transpose of its inverse, that is

$$[(I - A)']^{-1} = [(I - A)^{-1}]'$$

hence equation (3.9) may be written as

$$P = [(I - A)^{-1}]'(\pi b) \tag{3.10}$$

which is the required result as in (3.3), or alternatively by transposing (3.10) we obtain

$$P' = (\pi b)'(I - A)^{-1} \tag{3.11}$$

as in (3.4).

Example of Price Effects

An example based on the Irish 3-Sector 1960 model shows how the method works in practice. The technical and interdependence coefficients for this model are rewritten in Table 3.10.

TABLE 3.10

	Technical Coefficients			Interdependence Coefficients		
	Agriculture (1)	Industry (2)	Services (3)	Agriculture (1)	Industry (2)	Services (3)
Agriculture (1)	0·0109	0·1518	0·0038	1·0394	0·1946	0·0218
Industry (2)	0·1383	0·1822	0·0845	0·1833	1·2652	0·1150
Services (3)	0·0550	0·0599	0·0647	0·0729	0·0925	1·0778
Imports	0·0763	0·2227	0·0261			
Indirect Taxes less Subsidies	0·0212	0·0806	0·0083			
Income Arising	0·6668	0·2795	0·7619			
Depreciation	0·0314	0·0232	0·0508			
Total Inputs via Column Sum	0·9999	0·9999	1·0001			

Suppose we wish to increase the income arising in the agricultural sector by 5 per cent, and that in industry and services by 10 per cent. We assume that if this happens, imports in all sectors will increase by 4 per cent. If these increases come about with no change in the quantities of goods produced or in the manner of disposal, what output price increases must be obtained to give the desired results?* Let the output price increases be P_1, P_2 and P_3 for industries (1), (2) and (3) respectively, which means that the price increases for the outputs of each industry are the same whatever the purchasing sector, i.e. every entry in any row of the basic table is increased by the same amount. From formula (3.10)

$$P = [(I-A)^{-1}]' \begin{bmatrix} (0{\cdot}04 \times 0{\cdot}0763) + (0{\cdot}05 \times 0{\cdot}6668) \\ (0{\cdot}04 \times 0{\cdot}2227) + (0{\cdot}10 \times 0{\cdot}2795) \\ (0{\cdot}04 \times 0{\cdot}0261) + (0{\cdot}10 \times 0{\cdot}7619) \end{bmatrix} \qquad (3.12)$$

Simplifying the terms on the right and entering the numerical values for $[(I-A)^{-1}]'$ we obtain

$$\begin{matrix} P_1 = \\ P_2 = \\ P_3 = \end{matrix} \begin{bmatrix} 1{\cdot}0394 & 0{\cdot}1833 & 0{\cdot}0729 \\ 0{\cdot}1946 & 1{\cdot}2652 & 0{\cdot}0925 \\ 0{\cdot}0218 & 0{\cdot}1150 & 1{\cdot}0778 \end{bmatrix} \begin{bmatrix} 0{\cdot}0364 \\ 0{\cdot}0369 \\ 0{\cdot}0772 \end{bmatrix} \qquad (3.13)$$

* For implications of these changes see footnote on page 53.

Multiplying out as follows we obtain

$$(1{\cdot}0394 \times 0{\cdot}0364) + (0{\cdot}1833 \times 0{\cdot}0369) + (0{\cdot}0729 \times 0{\cdot}0772) = 0{\cdot}0502 = P_1$$
$$(0{\cdot}1946 \times 0{\cdot}0364) + (1{\cdot}2652 \times 0{\cdot}0369) + (0{\cdot}0925 \times 0{\cdot}0772) = 0{\cdot}0609 = P_2$$
$$(0{\cdot}0218 \times 0{\cdot}0364) + (0{\cdot}1150 \times 0{\cdot}0369) + (1{\cdot}0778 \times 0{\cdot}0772) = 0{\cdot}0882 = P_3$$

Hence to obtain the given increase in incomes, agricultural prices would need to increase by about 5 per cent, industrial prices by about 6 per cent and the price of services by about 9 per cent. The increase in the overall price level can also be calculated, if required, by weighting the various price increases by the amounts sold on the domestic market. Thus from Table 2.1 it can be seen that household and government consumption plus capital formation for the three sectors were: Agriculture 65·585; Industry 283·639 and Services 196·472. Multiplying these figures by the respective price increases in the different sectors and dividing by the total value of domestic consumption we obtain:

$$\frac{(65{\cdot}585 \times 0{\cdot}0502) + (283{\cdot}639 \times 0{\cdot}0609) + (196{\cdot}472 \times 0{\cdot}0882)}{65{\cdot}585 + 283{\cdot}639 + 196{\cdot}472}$$

$$= \frac{37{\cdot}8948}{545{\cdot}696} = 0{\cdot}0694$$

indicating an overall price increase of about 7 per cent carried by the combined outputs of the three domestic sectors. For these combined outputs to final demand of households alone, the price increase is 6·8 per cent.

If it is assumed that at the new prices, the same volume of exports as heretofore can be sold by industries (1), (2) and (3), then the price increase for the aggregate 3-sector exports is given by

$$\frac{(49{\cdot}750 \times 0{\cdot}0502 + 103{\cdot}278 \times 0{\cdot}0609 + 42{\cdot}090 \times 0{\cdot}0882)}{(49{\cdot}750 + 103{\cdot}278 + 42{\cdot}090)}$$

which is 0·0641.

Thus a contribution of 6·41 per cent of the original 3-sector export value is paid by foreign countries, towards the specified income increases. The amount in question is an extra £12·507 million. For an assumed 4 per cent increase in the £142·991 million imports to the three productive sectors, there is an extra import cost, to the domestic economy, of £5·720 million. Thus the net gain from foreign sources is £6·787 million.

If on the other hand the exports have to be sold at the original prices, a subsidy of £12·507 million would have to be provided by the home government, to cover the extra £12·507 million imputed to exports by the system. This, together with the £5·720 million extra cost of imports, would mean an extra £18·2 million having to be found from domestic sources to permit the specified income increases, for the stated export–import conditions.

It is to be noted that the 4-sector inter-industry structure and interdependence coefficients given in Tables 3.5 and 3.6 cannot be used for price calculations of this kind which imply a direct change of the coefficients in Row (4) – Household Income. Our approach to price changes given by formulae (3.9) or (3.10) depends upon these price increases being originated by changes in rows defined to be outside the inter-industry matrix, i.e. in primary input rows. In the numerical example the price changes are due to specified changes in income arising and hence the Household row and its corresponding column (Household Consumption) must be kept outside the inter-industry quadrant.

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EXERCISES

Exercise 3.1. If final demand for the products of Sector (1) in exercise (2.2) increases by one unit with no change in the final demands of the other sectors,

calculate the effect this will have on:

(a) Income arising in each of the different sectors and in all sectors combined,
(b) Total primary inputs of each sector and in all sectors combined,
(c) Imports of each sector and imports of all sectors combined,
(d) Subsidies of the different sectors and total subsidies.

Exercise 3.2. If indirect taxes are increased by 5 per cent and imports by 7 per cent, use the results from exercises 2.2 to show the percentage increase in prices of (a) the different sectoral outputs, and (b) the overall price level so as to leave the levels of the other primary inputs unchanged.

Section 2

Review of Input–Output Methodology

4 BASIC ASSUMPTIONS AND TREATMENT OF SPECIFIC ITEMS

Basic Assumptions

The static input–output system in its simplest form is founded on three assumptions.[2]

(i) Each sector produces a single output with a single input structure and there is no substitution between the outputs of different sectors;

(ii) The inputs into each sector are simple proportions only of the level of output of that sector, i.e. the amount of each kind of input absorbed by any particular sector goes up or down in direct proportion to the increase or decrease in that sector's total output;

(iii) The total effect of carrying out production in several sectors is the sum of the separate effects.

For a more detailed description see United Nations, Series F, No. 14.[1]

These assumptions are very important and should be kept in mind in both the construction and analysis of input–output tables. Here however we will pay particular attention to the first, known as the homogeneity assumption, because it explains why certain classifications are adopted in the construction of input–output transactions tables.

The homogeneity assumption requires (a) that all products of a single sector should either be perfect substitutes for one another or they should be produced in strictly fixed proportions, (b) that each sector should have a single input structure, and (c) that there should be no substitution between the products of different sectors. In other words the same product or a close substitute should not be included in two different sectors.

Unfortunately, these assumptions are not always compatible and indeed they are sometimes contradictory. For example, wool and synthetic fibres are close substitutes in consumption, hence in order to comply with the "no substitution" assumption they should be combined in the same sector. But they have completely different input structures, and if combined the principle of a "single input structure" is violated. What then should be done? Each case must of course be considered on its own merits but in general the rule is that where a choice such as this has to be made it is best to preserve the single-input structure rather than any other. In this case therefore wool and synthetic fibres should not be combined.

The degree of aggregation which should be adopted depends on many factors such as the purpose of the study, the availability of data, the time and

resources available for the study, etc. A detailed classification always provides more information than a highly aggregated one but, in general, the greater the degree of detail the greater the likelihood of substitution between sectors. Despite this difficulty, a commodity classification if it can be obtained is considered to be the most desirable, since each commodity has a single-input structure.

One serious problem, however, arises where a highly disaggregated commodity classification is adopted. The table becomes very unwieldy and it becomes exceedingly difficult to keep track of the various commodity flows. For this reason many workers consider it preferable to sacrifice a certain amount of homogeneity for the sake of brevity. This of course is a very personal matter. Some find it impossible to study a very large table, while others do not have the same difficulty in this respect.

Where commodities must be aggregated two classification criteria should be kept in mind. These are:

(i) Products which have a similar input structure should be grouped together even if they have different uses, e.g. malting barley and wheat; cars and tanks, and

(ii) Products whose outputs are likely to change in fixed proportions should be grouped together, e.g. cattle and dairying.

If products are classified in a sector so that neither of these conditions is fulfilled, then whenever the output levels of the constituent activities change, the inputs will not remain in the proportionate relation to output recorded in the base year.

Foreign Trade

In the preparation of an input–output table exports present no problem. They are treated as being part of final demand and are entered (usually at producer prices) in a separate column of this section of the table. (For description of producer prices see the section of the present Chapter denoted "Pricing")

The treatment of imports is more difficult however and requires special mention. In Table 1.1 all imports are entered in a primary input row. While this is a common way of treating imports it has the disadvantage of concealing the nature of the imports and for this reason other methods of dealing with them are often used. [2] In appraising these methods it is helpful to make a distinction between what are called competitive and non-competitive imports.

A competitive import can be defined as a commodity which is a close substitute for one domestically produced. A non-competitive import on the other hand is one for which there is no domestic counterpart. As an example of this distinction for the Irish economy we might classify maize imports as competitive (versus native barley) and tobacco as being non-competitive. Distinctions of this kind can, however, in extreme cases clash with common-sense reasoning

about how an imported commodity should be classified. Since a certain amount of hard wheat is necessary for the making of bread flour, much though not all of the imported wheat is non-competitive (being unlike the native soft wheat). There is a possible over-precise interpretation of the competitive definition in distinguishing between competitive and non-competitive wheat imports and in splitting a single commodity such as wheat, with a single end use (the production of flour), into two separate kinds. Accordingly a commodity group (such as hard wheat and soft wheat) is usually put in one classification or the other, depending on the category into which the majority is thought to fall.

If all imports are treated as non-competitive, their values are entered along a primary input row as is done for convenience in Table 1.1. In this case all intermediate rows must be of domestic products only, so that the construction of the table requires a breakdown of each commodity input, to show the values of the domestic and imported components used in each sector. For many items the true division of input between domestic and imported components is unknown. For example, under Irish conditions we do not know for sure how many of the imported cattle go for direct slaughter and how many go for further fattening on farms. The same is true for imported sheep and lambs and many other items. Thus the complete separation and valuation of import inputs to each sector presents problems of how to split commodities into domestic and imported components.

If some imports are classed as competitive their values should be entered in a special column with negative signs, to the right of total final demand. In this case the inter-industry flows contain both domestic and imported produce and this treatment improves the stability of the input coefficients and does not require detailed breakdown of such coefficients between domestic and imported components. The other imports, classed as non-competitive, have to be allocated as inputs to sectors and these allocations are usually aggregated into a single row of inputs. In this case however, it is necessary to specify in advance the column vector of values of the competing imports, as a part of the solution which it is desired to compute, in any application of this model to input–output calculations.

Though various other methods are available for incorporating imports into the model, probably the best solution is to enter the non-competitive imports in a special row and the competitive imports in a special column, as described in the previous paragraph. The competitive imports are items such as livestock, wool, textile piece goods, and the non-competitive (for Irish conditions, at least) are items such as raw cotton, citrus fruits, crude petroleum and most metallic components of vehicles and machinery.

Making Adjustments for Competitive Imports

In discussing the application of input–output methods in the previous

chapters it was shown that the output levels for the different sectors could be obtained by multiplying the vector of final demands by the interdependence coefficients, (i.e. $(I-A)^{-1}Y = X$). In the example given, however, all inter-sectoral flows were treated as being of domestic origin because in Table 1.1 all imports were classed as non-competitive and were included in a primary input row.

Where imports are treated as being competitive and are included in a separate column the inter-sectoral flows contain both imported and domestic produce. In such cases adjustments have to be made for the imported produce in determining the proper level of output from the system, since by definition output has imported items deducted. These adjustments are made by subtracting the values of competitive imports of each row from the final demands of that row before multiplying the vector so obtained by the interdependence coefficients thus:

$$(I-A)^{-1}(Y-M) = X \tag{4.1}$$

where $(I-A)^{-1}$ is the matrix of interdependence coefficients, X is a vector of outputs and $(Y-M)$ is a vector obtained by deducting from the total final demand of each sector, (y_i), the value of imports which are competitive with the output of that sector, (m_i).

The fact that imports have to be deducted in advance sometimes leads to difficulties in specifying final demands for planning purposes. For example if we specify certain levels of final demand for the various livestock sectors in an agricultural input–output model, how do we specify the levels of domestic and imported feeds consistent with these demands without having to do a lot of figuring outside the system? The best method is to specify initially zero levels of imports for the feed sectors which in effect means that all feed will be supplied from domestic sources. The various levels of final demand specified (including those for livestock) are then multiplied by the inverse matrix to determine the output levels consistent with the specifications. By applying the technical coefficients to these outputs we obtain the internal flows within the system which include the total amounts of the different feeds required for the levels of livestock postulated. We then estimate the amounts of these feeds which are likely to be domestically produced and specify that the balance will come from imports. The imports are then entered as negative quantities in the imports column and a further iteration carried out to obtain the required result. The above approach can be tedious and for that reason some workers prefer to manipulate the model so that the competitive imports are incorporated in the diagonal elements of the inverse matrix. When this is done the level of imports can be derived from the model and need not be predetermined. This treatment implies that competitive imports included in a row form a fixed proportion of the total flow along that row.

The point to be kept in mind, however, with this procedure is that the competitive imports derived from the model for any sector will always be in the same proportion to the sector's output as they were in the base year. This is not always a realistic assumption since for many commodities changes occur over time in the proportion of domestic supplies to competing imports. This difficulty can be partly overcome by incorporating some of the competitive imports in the inverse matrix and leaving others out. In an agricultural input–output model imports of live animals present few specification difficulties and may be left alone, while feedgrains which present many problems may be treated as described. The division of imports in this way presents no mathematical difficulties and can be recommended for certain purposes.

The method of incorporating a column vector of competitive imports in the inverse matrix is explained below.

Suppose the basic transactions table is as follows, where the x_{ij}'s contain both domestic produce and competitive imports.

Table 4.1

Industry	Intermediate Demand (1)	Intermediate Demand (2)	Final Demand	Less Competitive Imports	Domestic Output
1	x_{11}	x_{12}	Y_1	$-M_1$	X_1
2	x_{21}	x_{22}	Y_2	$-M_2$	X_2
Primary Inputs	Z_1	Z_2	–	–	–
Total Inputs	X_1	X_2	–	–	–

The technical coefficients for this system are shown in Table 4.2.

Table 4.2

Industry	Intermediate Demand (1)	Intermediate Demand (2)	Final Demand	Less Competitive Imports
	Technical Coefficients		Values	Ratios
1	a_{11}	a_{12}	Y_1	$-m_1$
2	a_{21}	a_{22}	Y_2	$-m_2$
Primary Inputs	z_1	z_2	–	–

Here $a_{ij} = \dfrac{x_{ij}}{X_j}$; $z_j = \dfrac{Z_j}{X_j}$ and

$$m_i = M_i/X_i \quad (\text{i.e. } m_1 = M_1/X_1; \quad m_2 = M_2/X_2)$$

The following system of equations can be written from the data in the above table

$$\begin{aligned} X_1 &= a_{11}X_1 + a_{12}X_2 + Y_1 - m_1X_1 \\ X_2 &= a_{21}X_1 + a_{22}X_2 + Y_2 - m_2X_2 \end{aligned} \tag{4.2}$$

Transferring all the X's to the left hand side and regrouping we obtain:

$$\begin{aligned} (1 - a_{11} + m_1)X_1 - a_{12}X_2 &= Y_1 \\ -a_{21}X_1 + (1 - a_{22} + m_2)X_2 &= Y_2 \end{aligned} \tag{4.3}$$

In matrix notation (4.3) can be written as

$$(I - A + M)X = Y \text{ so that} \tag{4.4}$$

$$X = (I - A + M)^{-1}\, Y \tag{4.5}$$

where $(I - A + M)$ is a matrix formed by adding to each diagonal element of the conventional $(I - A)$ matrix the competitive import ratio in the same row, these ratios being obtained by dividing the value of imports in each row by the output value of that row (i.e. the row total value of domestic output). X is the vector of outputs of domestic sectors and Y is the vector of final demands which may contain competitive imports as well as domestic produce.

It might be mentioned for completeness that competitive imports may be entered as a row vector in the primary input quadrant and by a somewhat similar manipulation incorporated in the diagonal elements of the inverse matrix also. The method of doing this is not discussed here but interested readers are referred to McGilvray [3] for a critical discussion of this and other methods of treating imports.

Pricing

In theory, the entries in a transactions table may be recorded in physical units. In practice, this cannot be done because physical quantities are not available for many items. Even if they were, however, commodities could not be aggregated in such units. For example, it would not be meaningful to add tons of potatoes to gallons of milk or to numbers of cattle or sheep. Therefore, the flows in a table are usually expressed in money values, these values being obtained by multiplying physical quantities by prices. It is useful, however, for statistical purposes, to prepare a transactions table for selected commodities in physical quantities. Such a table shows how the total production of important commodities is disposed of and brings to light discrepancies in the available

statistical data. Of course a transactions table in value terms is very useful for this purpose also and it is well worth preparing such a table for statistical purposes alone. Indeed in the early days of input–output, transactions tables were produced in many countries with the objective of providing a consistent framework for organizing economic statistics.

Any transaction if described in monetary rather than physical units may be valued at either the price received by the producer or at the price paid by the consumer. The difference between these two values is the marketing costs which include such items as transport costs, wholesale and retail mark-ups, insurance and warehouse costs, and net indirect taxes, i.e. indirect taxes less subsidies.

If the values are entered at producers' prices,* marketing costs must be shown either as inputs from the appropriate sectors or in a single row in the primary input section entitled "trade and transport margins". An example will explain why this must be done. Suppose we are constructing a row and column for the agricultural sector of an economy, using producers' prices. If we ignore for the moment all the unsold produce and imports, the entries across the row will be at prices received by farmers for products sold, and the total of the row will be the output of the agricultural sector. The entries in the agricultural column will be the values, at producers' prices also, of the inputs purchased by farmers. For example the inputs of animal feed will be valued f.o.b. feed compounders' stores. This value will be lower than that paid by farmers for the feed since the latter will contain a fairly substantial element for retailers' margins and for transport costs from compounders' stores to farmers' premises. Now since the row total must be equal to the column total the balancing item which is the "Income arising in agriculture" will be inflated unless the transport and trade margins are entered somewhere else in the column (they are normally included as a separate entry in a transport and trade row).

If, on the other hand, the entries are valued at purchasers' prices the row total of each sector includes the marketing costs incurred in each delivery. Row and column totals are therefore higher under this system of pricing than if producers' prices were used. If, however, "Income arising" is to be kept the same under both pricing systems the marketing costs must be double counted in the columns. This can be explained symbolically as follows:

If	Output at producers' prices	$= O$
	Input at producers' prices	$= N$
	Trade and Transport margins on output	$= T_o$
	Trade and Transport margins on input	$= T_n$

* The United Nations' recommended system of valuation (see references [14] and [15] of the Introduction) includes commodity taxes on outputs as part of the U.N. definition of "Producers' Values". The system of pricing which they denote "Basic Values, Approximate" corresponds to values at producers' prices, as defined in this text. See the "Glossary of Main Terms" in [15] for details of U.N. definition.

Income arising $= I$

Output at purchasers' prices $= O + T_o$

and

Inputs at purchasers' prices $= N + T_n$

Then at producers' prices, and ignoring imports,

(a) $$I = O - (N + T_n)$$

while at purchasers' prices, if I is to be the same as in (a),

(b) $$I = (O + T_o) - (N + T_n + T_o)$$

T_o is gathered together as one single unit in a transport and trade, or distribution sector whereas T_n will be combined (in pieces) with the various inputs and will not appear as such. For this reason, and because transport and trade margins will probably vary with output distribution changes, valuation at producers' prices is usually favoured for all products sold, including exports. Under this valuation system, however, care must be taken in interpreting the figures in the export column. It needs to be kept in mind that the values in this column are not the usual f.o.b. export values. They will be higher or lower depending on the item involved.

If the price received by producers for a commodity contains a subsidy element, the official export value of this commodity or of a product made from it (e.g. pigs and bacon) will normally be lower than the export value given in the input–output table. On the other hand if the value of a commodity does not contain a subsidy element, the official export value will likely be slightly higher than that in the Input–Output table because of transport charges between producers' premises and point of embarkation.

Valuing Imports

If imports are valued at c.i.f. (import) prices, the trade and transport margins which arise on the imported goods give rise to no complications. They are simply included in a distribution or trade row elsewhere in the table. If, on the other hand, imports are valued at purchasers' prices, it is necessary to enter the trade and transport margins on them as a negative entry in the distribution or trade column of the import row. This gives the correct c.i.f. import total for the import row but gives a peculiar column for distribution or trade. For this reason it is best to value imports at c.i.f. (import prices).

Treatment of Subsidies and Indirect Taxes

Standard procedures are available for the treatment of subsidies in input–output analysis but seldom can these rules be applied in a straightforward manner to all products because of the many peculiar ways in which commodities are subsidized.

Subsidies paid direct to producers can be dealt with simply enough. Here the unsubsidized market value received by producers is entered in the relevant row and the subsidy entered with a minus sign in the producers' column along a special subsidy row. The minus entry has the effect of increasing the residual item "income arising" in the producing sector concerned.

Subsidies on primary inputs can also be dealt with fairly simply. Two methods are available. When the price of an input is reduced due to a subsidy payment, the low subsidized price may be entered in a primary input row and the subsidy ignored. This treatment gives the correct level of income for the sector involved but it does not show the magnitude of the subsidy. Hence, if the total amount of subsidies paid by the Government are to be shown in an input–output table, subsidized inputs must be entered at unsubsidized prices (i.e. at prices paid plus subsidy) and the subsidies entered with minus signs in the subsidy row.

Subsidies paid by means of guaranteed prices are usually more difficult to deal with and different methods have to be adopted in different situations. As a general rule in such cases the guaranteed price is normally used in valuing a subsidized product in the row of the producing sector. This subsidized value will therefore appear as a cost in the column of the purchasing sector. Purchasers, however, may have to sell the commodity concerned (or products made from it) at a lower value than that at which purchased, and unless some adjustment is made an apparent loss will be incurred on the transaction. To counteract this loss the subsidy should normally be entered in the subsidy row and distributed with a minus sign to the column of the purchasing sector.

Indirect taxes (also denoted Taxes on Expenditure) may be treated in different ways, but at producers' prices the commodity values entered across the inter-industry rows are the amounts which would be obtained by sellers for the products if they were not taxed. The taxes are then entered in a special primary input row and distributed with positive signs to the sectors using the taxed products. Thus, petrol is sold to industries at the low untaxed price but the tax is entered as a further cost to these industries in the indirect tax row. Similarly, tobacco, drink, and petrol are sold to households at untaxed prices but the tax is entered further down the household column in the indirect tax row.

Customs duties are treated in exactly the same way as indirect taxes. The values entered in the import column or row as the case may be, are the import values before addition of any duties. The duties are charged to the industries utilizing the products by the inclusion of a primary input row for such duties Very often customs and excise duties are put in the same row as indirect taxes.

Valuation of Unsold Produce

In official statistics unsold produce consumed by persons is sometimes valued

at production cost but more usually valued at prices received for similar produce sold. It is reasonable to use this system of pricing for such produce in preparing input–output tables.

For produce fed to animals but not coming onto the market, the use of producer market prices is not entirely realistic. For example, only about one-third of the Irish potato crop is sold, the remainder being fed to farm animals on the farms where produced, or going to waste. All the potatoes in the State could not be sold at the average market price ruling in any year and, indeed, if farmers were to value potatoes fed to animals at market prices, the animals fed would likely show a substantial loss. For this reason products of this kind are often valued at cost-of-production prices in the preparation of input–output tables. For a method of calculating prices for such products see, for example, O'Connor and Breslin.[4] It might be mentioned however that this procedure makes for certain difficulties if the model is used for planning purposes. Hence some workers prefer to use the same price for a commodity regardless of its use.

REFERENCES – CHAPTER 4

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[2] McGilvray, J., "The Stability of Coefficients in an Irish Inter-Industry Model", *Journal of the Statistical and Social Inquiry Society of Ireland*, Vol. XXI, Part III (1964–5).

[3] *ibid.*, p. 61.

[4] O'Connor, R., with Breslin, M., "An Input–Output Analysis of the Agricultural Sector of the Irish Economy", *Paper No. 43 of the Economic and Social Research Institute*, Dublin (1968).

EXERCISES

Exercise 4.1 The following is an undifferentiated - imports transactions table i.e. all imports are included in a primary input row and all entries in the intermediate and final cells relate to home produce only.

TABLE A

Sector	(1)	(2)	Final Demand	Total Output
1	5	10	15	30
2	4	6	30	40
Total Imports	9	8	6	23
Other Inputs	12	16	2	30
Total Inputs	30	40	53	123

The competitive imports are given in tabular form as follows:

TABLE B

Sector	(1)	(2)	Final	Total
1	3	4	1	8
2	2	2	3	7
Total competitive imports	5	6	4	15

(a) Construct the differentiated-imports transactions table in the following form.

TABLE C

Sector	(1)	(2)	Final Demand	Less Competitive Imports	Total Output
1 2					
Non Competitive Imports Other Inputs					
Total Inputs					

(b) Calculate technical and interdependence coefficients for Tables A and C above and show that the same outputs are obtained when the respective interdependence coefficients are applied to final demands in Table A, and to final demands less competitive imports in Table C.

(c) Incorporate the competitive imports in the diagonals of Table C as explained in equation (4.3) above. Invert the $(I - A + M)$ matrix so obtained. Post-multiply this inverse by the vector of final demand in Table C which includes imports, and note that the result gives the correct vector of outputs.

Exercise 4.2 Recast the following Table A, expressed in producers' prices, into purchasers' prices form.

Explanation

It costs 4 for distribution to supply agriculture with its inputs. Of this amount the distribution costs of providing grain milling inputs is 3 and of providing sugar refining inputs is 1. Similarly for other inputs. For simplicity it is assumed that the distribution industry uses no material inputs.

TABLE A

	Agriculture	Grain Milling	Sugar Refining	Distribution	Final Demand	Total Output
Agriculture	–	10	8	0	18	36
Grain milling	6	–	2	0	16	24
Sugar refining	3 (3)	2 (5)	– (4)	0	17 (6)	22
Distribution	4 (1)	6 (1)	5 (1)	–	13 (5) (2)	28
Primary Inputs	23	6	7	28	30	94
Total Inputs	36	24	22	28	94	204

Partial Solution

A partial solution to the problem is given in the following Table B. Complete the remaining blank cells of the table and note that the different outputs are higher than they were in the previous table.

TABLE B

	Agriculture	Grain Milling	Sugar Refining	Distribution	Final Demand	Total Output
Agriculture	–	(10) 15 (5)	(8) 12 (4)	0	(18) 24 (6)	51
Grain milling	(6) 9 (3)					
Sugar refining	(3) 4 (1)					
Distribution	(5) 15 (4) (6)					
Primary Inputs	23					
Total Inputs	51					

5 TREATMENT OF SECONDARY AND JOINT PRODUCTS

Description

The establishment or firm is the basic unit in the industrial statistics of most countries but as is well known many establishments produce more than one product. Usually one of the products is of primary importance and the others are secondary. For example cattle slaughtering plants produce beef, hides, fats and offals, beef being the primary product and the others being of secondary importance. Similarly, flour millers produce flour, bran, pollard and compound feeding-stuffs for animals, flour being primary and the others secondary.

Several classes of secondary products can be distinguished but strictly speaking only those classes of secondary products whose production is independent of the primary product should be referred to as secondary. Products which are the output of a single technical process fall into the category of joint products. Examples of secondary products are compound feeds, produced by flour millers, grass meal produced by the Irish Sugar Company, potato crisps produced by a tobacco company and so on. These products have separate input structures and the derivation of independent cost figures for them presents no conceptual difficulties. Examples of joint products are flour and bran, beef and hides, mutton and wool, malt and combings, etc. The general characteristics of these products is that they are the output of one production process and so share a common input structure. Consequently, they cannot be costed separately. Also the supply of one cannot usually be increased without a corresponding increase in the other. Hence they are produced in strict proportion.

Both secondary and joint products give rise to considerable difficulties in input–output analysis. The main problem is that the allocation to one sector, of an establishment having several products, impairs the principles of sector homogeneity and gives rise to misleading results when the table is used for planning purposes. For example, since the cattle slaughtering industry produces beef and hides in more or less fixed proportions an increase in the consumption or exports of beef will lead to an increase in production and slaughtering of cattle and to the production of hides. An increase in the consumption of hides on the other hand does not automatically lead to an increase in the production of cattle or the consumption and export of beef, though this is what a conventional input–output table would show. Such a table would also show that if final demand for leather were postulated to increase by some given amount

while that of meat were assumed to remain constant a certain quantity of meat would be routed to the tanning industry to supply the demand for hides. A similar thing could happen with mutton and wool or indeed with any other secondary or joint-product situation.

A number of methods are available for dealing with problems of this kind depending on the nature of the products concerned. The most satisfactory solution which applies to all classes of secondary products (other than joint products) is to separate the inputs used in the production of the secondary products from those used in the production of the primary products and to rearrange, in one sector, all products of a given type regardless of where they have been produced. This procedure is known as *Redefinition* and is always employed to a greater or lesser extent in preparing transactions tables. As indicated above, redefinition cannot be carried out in a satisfactory manner for joint products, and for these, other techniques have to be employed for separating them from one another. Such techniques can also be used for secondary products if sufficient data to enable redefinition are not available.

Artificial Entries

If joint products are distributed to consuming sectors along the same row they are implicitly assumed to be the same product and in subsequent manipulations of the table they are treated as such, often with misleading consequences. If, however, they are distributed along different rows to the consuming sectors they will be treated as separate products in subsequent operations. Distribution along separate rows can be performed by the introduction of artificial entries to the original table as follows:

Case I. If the secondary or joint product is a commodity for which there is already a sector in the table in which it can be entered, the product is first sold to this sector and from there distributed to the sectors using it. Two methods of doing this are available.

(a) When the output of the secondary or joint product varies in proportion to the output of the producing sector (which is what usually happens in practice), the secondary product is transferred from the producing Sector A to an appropriate Sector *B* and distributed to the consuming Sector *C* along the *B* row as shown in the following example.

Suppose we have three sectors in a Transactions Table, i.e. brewing, animal feed, and livestock. Suppose also that the brewing industry produces malt combings to the value of £170 thousand which are sold directly to farmers for livestock feeding. In the initial construction of a Transactions Table we would show the figure of 170 in the livestock column of the brewing row as shown in Table 5.1. If however we decide on the more sophisticated procedure the table would be adjusted, as shown in Table 5.2.

TABLE 5.1

	Brewing	Animal Feed	Livestock	Total
	A	*B*	*C*	
Brewing *A*	–	–	170	170
Animal Feed *B*	–	–	–	–
Livestock *C*	–	–	–	–
Total	–	–	170	170

TABLE 5.2

	Brewing	Animal Feed	Livestock	Total
	A	*B*	*C*	
Brewing *A*	170	–	–	170
Animal Feed *B*	–170	–	170	–
Livestock *C*	–	–	–	–
Total	–	–	170	170

In the first case the malt combings are sold directly to the livestock sector. In the second case they are shown first as being used in the brewing sector by means of the positive entry in the brewing row and column. They are then shown as being sold to the animal feed sector (by means of the negative entry in the brewing column of the animal feed row) from which they are distributed along this row to the livestock column. As can be seen, the procedure leaves unchanged the row and column totals. In both cases the total of the brewing row and the livestock column is 170.

(b) When the output of the secondary product varies with the input of the sector to which it is to be transferred, a somewhat different procedure is adopted which is explained in the example given below. Cases such as this do not arise with joint products (only with secondary products) but if we assume for expository purposes that in the above example the output of malt combings varied in proportion to the animal feed industry, the adjusted table would appear as follows:

TABLE 5.3

	Brewing	Animal Feed	Livestock	Total
	A	*B*	*C*	
Brewing *A*	–	170	–	170
Animal Feed *B*	–	–170	170	–
Livestock *C*	–	–	–	–
Total	–	–	170	170

As can be seen from Table 5.3 the malt combings in this case are sold directly to the animal-feed industry by means of a positive entry in the animal-feed column of the brewing row. They are then transferred to the animal-feed row by means of an offsetting negative entry in the diagonal cell of the animal-feed sector, from whence they are distributed along this row to the livestock column. As in the previous case the procedure leaves the row and column totals the same as they were in the original table (Table 5.1).

Case II. If the secondary or joint product is one which cannot be entered in any of the sectors of the table and if it cannot be classified into a sector of its own because separate input data for it are not available or cannot be defined, the practice is to enter it in an artificial sector which represents no real production but merely permits distribution to consuming sectors. The method of doing this is as follows.

The value of the joint product should be entered with a positive sign in the diagonal cell of the producing sector and with a negative sign in the producing sector column of a new artificial row. In this way, the joint product is transferred to the artificial row along which it is distributed to the consuming sectors. An artificial column is also introduced so that the matrix will have the same number of rows as columns. The column, however, is left blank. An example using beef and hides will explain the method clearly.

Because beef and hides have a common input structure the allocation of separate costs to one product or the other is not feasible, hence redefinition is impossible. A real sector for hides cannot therefore be defined and so an artificial sector must be introduced. Let us assume, for expository purposes, that the value of beef output from cattle slaughter is £30 million and that of hides is £2 million. We assume also that all of the beef goes to final demand while all the hides go to the tanning industry. Table 5.4 shows the relevant section of the Transactions Table as it would appear if no adjustment for joint products were made.

TABLE 5.4

	Cattle Slaughter	Tanning	Final Demand	Total Output
Cattle Slaughter	–	2	30	32
Tanning	–	–	3	3
Primary Inputs	32	1		
Total Inputs	32	3		

The adjustment is made by including an artificial row and column for hides as shown in Table 5.5.

TABLE 5.5

	Cattle Slaughter	Hides	Tanning	Final Demand	Total Output
Cattle Slaughter	2	–	–	30	32
Hides	−2	–	2	–	–
Tanning	–	–	–	3	3
Primary Inputs	32	–	1		
Total Inputs	32	–	3		

In Table 5.4 the hides are sold directly to the tanning industry while in Table 5.5 they are shown as being first used in the cattle-slaughter industry by means of the positive entry in the diagonal cell of that sector. They are then shown as being sold to the artificial-hides sector by means of the negative entry in the cattle-slaughter column of the hides row, from whence they are distributed to the tanning column. As can be seen the procedure leaves unchanged the row and column totals. The artificial-hides sector has counterbalancing entries of −2 and +2 in the row giving zero for the row total. The hides column has blanks in all cells.

Let us now examine, using both Table 5.4 and 5.5, the effects of an increase in the demand for leather (i.e. the product of the tanning industry) keeping in mind that due to technical factors the output of beef and hides from cattle slaughtering must remain in the same proportion as they were in the base year, i.e. in the ratio of $30/2 = 15/1$. Because there are very few figures in the tables concerned, this examination can be done without calculating technical or interdependence coefficients. In order to understand the discussion however, the following points should be kept in mind:

(1) A change in the final demand for the products of a sector directly affects the inputs of that sector, i.e. the sector's column entries. Each non-zero entry in this column will undergo a change proportional to the change in output which has taken place as a result of the change in final demand. Zero column entries will, of course, be unaffected.

(2) A change in final demand for the products of a sector does not directly affect any of the sector's other row entries, with the exception of the diagonal entry (if non zero) and the total output. The diagonal element is affected because it is a column as well as a row entry. Total output is affected because it is the row total.

(3) Since the column entries of any sector are elements in the rows of other sectors, changes in the entries of any column affect the total outputs of the relevant *real* rows. Other entries in these rows however will not be affected. We explain later what happens to artificial rows in such cases.

Effect of Change in Final Demand for Leather

Suppose the final demand for leather increases by say 3 units (£3 million) with no change in final demand for beef. Reference to the tanning row of Table 5.6 shows that this change will increase final demand for tanning to 6 units but since there is no entry in the diagonal cell of the tanning sector there will be no further increase in the output of tanning so that the revised tanning output will also be 6 units.

Reference to the tanning column of Table 5.4 shows that for every three units of total output of tanning 2 units of cattle slaughter and 1 unit of primary inputs are required. Therefore, six units of tanning output require 4 units of cattle slaughter and 2 units of primary inputs. The sum of these two figures brings the total inputs to 6 which is the same as the total output of the tanning sector, see Table 5.6.

Now the two extra units of cattle slaughter required by tanning will increase the output of cattle slaughter by 2 units to 34 units but since the output of cattle slaughter and hides must be in the proportion of 15/1 the amounts of these products in the new 34 units of output is 31·875 of beef and 2·125 of hides. Since, however, only 30 units of beef are required for final demand the remaining 1·875 units must have gone to the tanning industry to supply a demand for hides. This, of course, does not make sense. Table 5.6 which is based on the classification in Table 5.4 shows the position after the change in the tanning sector.

TABLE 5.6

	Cattle Slaughter	Tanning	Final Demand	Total Output
Cattle Slaughter	–	4 {1·875(B), 2·125(H)}	30(B)	34 {31·875(B), 2·125(H)}
Tanning	–	–	6 {1·875(B), 2·125(H), 2·00(VA)}	6
Primary Inputs	34	2		
Total Inputs	34	6		

B = Beef; H = Hides; VA = Value Added

Let us now use Table 5.5 to see what happens when final demand for leather is assumed to increase by 3 units and final demand for beef remains constant. As in the previous case the output of tanning will increase from three to six units, and so every entry in the tanning column of the table will be doubled,

i.e. total inputs will increase to 6, primary inputs to 2 and the input of hides to 4. Since there is no entry in the cattle slaughter row of this column, cattle slaughtering will not be affected by the increased demand for hides. Hence, output and inputs of cattle slaughter remain constant. The extra hides required, therefore, cannot come from domestic cattle slaughtered. They must come from some other source and of course the only other source of hides is imports. To show how imports can supply hides a competitive imports column is needed in the Table. Such a column is included in Table 5.7 which is based on Table 5.5 and shows the position after the change in the tanning sector. As can be seen, the import column of this table has an entry of −2 in the hides row. This counterbalances the extra two units of hides going to tanning and preserves the zero total for the artificial hides row.

TABLE 5.7

	Cattle Slaughter	Hides	Tanning	Final Demand Flow	Total Domestic	Imports	Total Output
Cattle Slaughter	2	–	–	30	32	–	32
Hides	−2	–	4	–	2	−2	–
Tanning	–	–	–	6	6	–	6
Primary Inputs	32	–	2				
Total Inputs	32	–	6				

THE SPECIFICATION OF FINAL DEMANDS FOR THE PRODUCTS OF ARTIFICIAL SECTORS WITH BLANK COLUMNS

It has been shown above that in planning an economy realistic and consistent final demands have to be specified for the different sectors.

In the case of an artificial sector with a blank column the row total (i.e. the total output) must always be zero, and the problem arises as to how the final demand for such a sector can be specified to give this result. A little reflection shows that this specification presents no real problems. An artificial sector with a blank column does not interact with any other sector and so we can specify in advance any level of final demand we wish, and determine the true level by multiplying the vector of final demands by the matrix of interdependence coefficients.

We explain the procedure using as an example the artificial-hides sector discussed previously. To do this, technical and interdependence coefficients are

calculated from the data in Table 5.7. These are given in Tables 5.8 and 5.9 respectively.

TABLE 5.8 Technical Coefficients based on Table 5.7

–	Cattle Slaughter x_1	Hides x_2	Tanning x_3
Cattle Slaughter	$\frac{1}{16}$	0	0
Hides	$-\frac{1}{16}$	0	$\frac{2}{3}$
Tanning	0	0	0
Primary Inputs	1·0	0	$\frac{1}{3}$
Total Inputs	1·0	0·0	1·0

TABLE 5.9 Interdependence Coefficients based on Table 5.7

	Cattle Slaughter x_1	Hides x_2	Tanning x_3
Cattle Slaughter	16/15	0	0
Hides	−1/15	1	2/3
Tanning	0	0	1

Example 1

Suppose, as in a previous example, we wish to increase the final demand for leather from the 3 units in Table 5.5 to 6 units with no change in the final demand (of 30 units) for beef. By assuming that under those conditions final demand for hides is Y_2 the outputs of the different sectors are obtained as follows:

$$\underset{(I-A)^{-1}}{\begin{bmatrix} 16/15 & 0 & 0 \\ -1/15 & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}} \underset{Y}{\begin{bmatrix} 30 \\ Y_2 \\ 6 \end{bmatrix}} = \underset{X}{\begin{bmatrix} 32 \\ Y_2 + 2 \\ 6 \end{bmatrix}} = \underset{X}{\begin{bmatrix} 32 \\ 0 \\ 6 \end{bmatrix}} \tag{5.1}$$

Here $(I - A)^{-1}$ is the matrix of interdependence coefficients which is multiplied by Y the vector of final demands to give X which is the vector of outputs. Each element of X is obtained by multiplying the elements of the corresponding row of the $(I - A)^{-1}$ matrix by the corresponding elements of the Y vector and totalling the results. Thus:

$$
\begin{aligned}
X_1 &= (\tfrac{16}{15} \times 30) + (0 \times Y_2) + (0 \times 6) = 32 \\
X_2 &= (-\tfrac{1}{15} \times 30) + (1 \times Y_2) + (\tfrac{2}{3} \times 6) = Y_2 + 2 \qquad (5.2) \\
X_3 &= (0 \times 30) + (0 \times Y_2) + (1 \times 6) = 6
\end{aligned}
$$

As can be seen, the output of the hides sector X_2 is $Y_2 + 2$, but since the output of this sector must be zero, $Y_2 + 2 = 0$, and so $Y_2 = -2$. Hence in order to satisfy our conditions, final demand for hides must be -2 units which, in fact, means an import of 2 units of hides. This level of final demand however need not be specified in advance. It can be determined from the exercise by including a symbolic value for it.

Example 2

Suppose as in a previous example the final demand for beef is specified to increase from 30 to 34 units with no change in the final demand for leather which is 3 units. What level of final demand for hides will leave the output of the hides sector zero?

Proceeding as in Example 1 and letting final demand for hides be Y_2 units we obtain:

$$
\begin{array}{ccccccc}
(I-A)^{-1} & & Y & & X & & X \\
\begin{bmatrix} \frac{16}{15} & 0 & 0 \\ -\frac{1}{15} & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix} & & \begin{bmatrix} 34 \\ Y_2 \\ 3 \end{bmatrix} & = & \begin{bmatrix} 36\frac{4}{15} \\ Y_2 - \frac{4}{15} \\ 3 \end{bmatrix} & = & \begin{bmatrix} 36\frac{4}{15} \\ 0 \\ 3 \end{bmatrix}
\end{array} \qquad (5.3)
$$

In this case the output of hides is $Y_2 - \frac{4}{15}$ but since this output must be zero, $Y_2 = \frac{4}{15}$. Hence in order to satisfy our conditions, the final demand for hides must be $\frac{4}{15}$ units (available for export, presumably).

As in the previous case this level of final demand need not be specified in advance. It can also be determined from the exercise by including a symbolic value for it.

In practical work one procedure adopted is to enter zero values for the final demands of all artificial sectors which have blank columns and determine the real values by solution of the matrix equation:

$$(I-A)^{-1}(Y-M) = X \qquad (5.4)$$

After completion of the calculations, the outputs of the artificial sectors are examined, when it will be found that they are either positive, negative or zero. These initial output values, in order to be adjusted to zero level, must therefore be balanced by a final demand entry equal in magnitude but of opposite algebraic sign. Hence, an initial zero output for an artificial sector indicates that final demand for the product of such a sector can be left at zero level.

A positive initial output requires an import of the same numerical magnitude, while a negative initial output requires export or home consumption of this magnitude. Thus in all cases the finalized row total will be zero.

EXERCISES

Exercise 5.1 The following input–output table has joint products across some of the rows.

TABLE A

	Sugar Beet	Cattle	Sugar Refining	Animal Slaughter	Final Demand	Total Output
Sugar Beet	–	3	10	–	–	13
Cattle	–	–	–	30	20	50
Sugar Refining	–	2	–	–	10	12
Animal Slaughter	–	3	–	–	(32) 34(2)	37
Primary Inputs	13	42	2	7	–	64
Total Input	13	50	12	37	64	176

The entries in this table are as follows:

Sugar beet – sells 3 units of beet tops as livestock feed and 10 units of roots to sugar refining.

Cattle – sells 30 units to animal slaughter and 20 to final demand (i.e. live exports).

Sugar refining – sells 10 units of sugar to final demand and 2 units of beet pulp for cattle feed.

Animal slaughter – sells 34 units to final demand of which 32 units are meat and 2 are hides. It also sells 3 units of meat and bone meal for animal feed.

(a) Prepare a new table to take these entries, having artificial sectors for cattle feed and hides. Include sugar beet tops, sugar pulp and meat and bone meal in the cattle-feed sector.

Result

TABLE B

	Sugar Beet	Cattle	Animal Feed	Sugar Refining	Animal Slaughter	Hides	Final	Total Output
Sugar Beet	3	–	–	10	–	–	–	13
Cattle	–	–	–	–	30	–	20	50
Animal Feed	–3	8	–	–2	–3	–	–	–
Sugar Refining	–	–	–	2	–	–	10	12
Animal Slaughter	–	–	–	–	5	–	32	37
Hides	–	–	–	–	–2	–	2	–
Primary Inputs	13	42	–	2	7	–	–	64
Total Inputs	13	50	–	12	37	–	64	176

(b) Prepare interdependence coefficients from the data in Table B.
(c) If final demands for cattle, sugar, and beef increase to the following amounts (cattle 25, sugar 12, and beef 36), what levels of final demand for animal feed and hides will leave the outputs of these latter sectors zero?

6 UPDATING OF COEFFICIENTS

Estimation of Inter-Industry Flows

As stated in Chapter 3, the RAS method devised by Stone and his associates at Cambridge University is one of the most commonly used techniques for making coefficient changes. An example based on our Irish 3 x 3 model for 1960 shows how the method, with slight modification, is applied. Suppose in 1968 we wish to make plans for the economy in 1972 and have available, at the time of planning, an input–output table for 1960 and conventional census of production and national income data for 1966. From the available data, the first operation is to prepare outputs, final demands and primary inputs for the different sectors in 1966 corresponding with these items in the 1960 input–output table. This is not a very difficult operation since some of these figures will be directly available in the required form in the 1966 national accounts data and the remainder can be estimated by disaggregation of other national-account entries using census of production data.

The total intermediate flows for each sector in the 1966 table are then calculated by deducting final demands from outputs to obtain the intermediate row totals, and by deducting primary inputs from outputs to obtain the intermediate column totals.

TABLE 6.1 Total for Intermediate and Final Sectors in 1966. £ million

Sectors	Inter-Industry				Final Demand	Total Output
	Agriculture	Industry	Services	Total		
Agriculture				110·769	138·594	249·363
Industry				286·497	548·920	935·417
Services				133·203	443·065	576·268
Total Intermediate	52·196	374·996	103·277	530·469	1 230·579	1 761·048
Primary Inputs	197·167	560·421	472·991	1 230·579	279·308	1 509·887
Total Inputs	249·363	935·417	576·268	1 761·048	1 509·887	3 270·935

Outputs, primary inputs, final and total intermediate demands for the Irish economy in 1966, prepared as explained above, are given in Table 6.1. The total intermediate demands for 1966 and 1960 are next used to prepare intermediate technical coefficients for 1966 by adjusting the 1960 transactions in the ratios

TABLE 6.2 Preparing a Matrix of Intermediate Transactions for the Irish Economy in 1966 Based on 1960 (3x3) Model

	INTERMEDIATE TRANSACTIONS			INTERMEDIATE TOTALS		RATIO b/a_i
	Agriculture	Industry	Services	1960	1966	$\hat{P}_1 = \frac{b}{a_1}$
	1960 figures ($\hat{T_o}$)			(a_1)	(b)	
1. Agriculture	2·180	81·687	1·143	85·010	110·769	1·30301 p_1
2. Industry	27·709	98·036	25·457	151·202	286·497	1·89480 p_2
3. Services	11·020	32·242	19·487	62·749	133·203	2·12279 p_3
4. Total	40·909	211·965	46·087	298·961	530·469	
		$\hat{P}_1 \hat{T_o} = T_1$				
5. Agriculture	2·84056	106·43898	1·48934	110·76888		
6. Industry	52·50301	185·75861	48·23592	286·49754		
7. Services	23·39315	68·44300	41·36681	133·20296		
8. Total	78·73672	360·64059	91·09207	530·46938		
9. Total 1966	52·196	374·996	103·277	530·469		
10. Ratio (9)/(8)	0·662918	1·039805	1·133765			
$\hat{Q}_1$	(q_1)	(q_2)	(q_3)			
		$T_1 \hat{Q}_1 = T_2$		(a_2)	(b)	$\hat{P}_2 = \frac{b}{a_2}$
11. Agriculture	1·883058	110·675784	1·688562	114·247404	110·769	0·969554 p'_1
12. Industry	34·805190	193·152731	54·688198	282·646119	286·497	1·013624 p'_2
13. Services	15·507740	71·167374	46·900241	133·575355	133·203	0·997212 p'_3
14. Total	52·195988	374·995889	103·277001	530·468878	530·469	

		$\hat{P}_2 T_2 = T_3$				
15. Agriculture	1·825726	107·306149	1·637152	110·769027		
16. Industry	35·279376	195·784244	55·433270	286·496890		
17. Services	15·464504	70·968959	46·769483	133·202946		
18. Total	52·569606	374·059352	103·839905	530·468863		
19. Total 1966	52·196	374·996	103·277	530·469		
20. Ratio $\frac{(19)}{(18)}$	0·992893	1·002504	0·994579			
$\hat{Q}_2$	(q'_1)	(q'_2) $T_3 \hat{Q}_2 = T_4$	(q'_3)	(a_3)	(b)	$\hat{P}_3 = \frac{b}{a_3}$
21. Agriculture	1·812751	107·574844	1·628277	111·015872	110·769	0·997776 p''_1
22. Industry	35·028645	196·274488	55·132766	286·435899	286·497	1·000213 p''_2
23. Services	15·354598	71·146665	46·515946	133·017209	133·203	1·001397 p''_3
24. Total	52·195994	374·995997	103·276989	530·468980	530·469	
		$\hat{P}_3 T_4 = T_5$		(a_4)		
25. Agriculture	1·808719	107·335598	1·624656	110·768973		
26. Industry	35·036106	196·316294	55·144509	286·496909		
27. Services	15·376048	71·246057	46·580929	133·203034		
28. Total	52·220873	374·897949	103·350094	530·468916		
29. Total 1966	52·196	374·996	103·277	530·469		
30. Total Output 1966	249·363	935·417	576·268	1 761·048		

of the total intermediate demands in the two years.* The method of carrying out these adjustments is shown in Table 6.2.

In the first section of this table we show the intermediate transactions for 1960 together with the total of these entries taken from Table 2.1. The corresponding row totals for 1966 taken from Table 6.1 are given in this section also, together with the ratios of the intermediate row totals in the two years. The matrix of intermediate transactions in 1960 is designated T_o while the column of row ratios is termed $\hat{P}_1$. The individual ratios in this column are termed p_1, p_2, p_3.

The second section of Table 6.2, i.e. T_1, is obtained by multiplying each row of the matrix of intermediate entries in the first section, T_o, by the corresponding element of the $\hat{P}_1$ ratio column. Thus the first row of T_o is multiplied by p_1, the second row by p_2 and the third row by p_3.

The first column of T_1 is obtained as follows

$$\begin{aligned} 2{\cdot}180 \times 1{\cdot}30301 &= 2{\cdot}84056 \\ 27{\cdot}709 \times 1{\cdot}89480 &= 52{\cdot}50301 \\ 11{\cdot}020 \times 2{\cdot}12279 &= 23{\cdot}39315 \end{aligned}$$

The second and third columns of T_1 are obtained in a similar manner. The effect of the above multiplications is to make the sum of the rows of T_1 equal to the corresponding row totals for 1966. When the columns of T_1 are added, however, the totals are not equal to the corresponding 1966 column totals (taken from Table 6.1) as can be seen by reference to Rows 8 and 9 of Table 6.2. The next operation therefore is to obtain ratios of these column totals by dividing the elements in Row 9 by the corresponding elements in Row 8 to obtain a row of ratios (Row 10) which we designate $\hat{Q}_1$ and having individual elements q_1, q_2 and q_3.

The columns of T_1 are now multiplied by the $\hat{Q}_1$ ratios to obtain the matrix in the third section of the table. The first column of T_1 is multiplied by q_1, the second by q_2 and the third by q_3 to obtain the elements of T_2. It will be seen that the total of the columns of T_2, given on line 14 are exactly the same as the 1966 totals given in line 9 but when we add across the rows of T_2 the totals are not the same as the corresponding 1966 figures, given in the adjacent column.

We continue therefore, by calculating a further column of P ratios which are designated as $\hat{P}_2$ and having individual elements p'_1, p'_2, and p'_3. These ratios are then multiplied by the corresponding rows of T_2 to obtain the matrix T_3, the totals of whose columns in Row 18 come fairly close to the corresponding 1966 entries in Row 19. We next divide Row 19 by Row 18 to obtain the ratios in Row 20 which we designate $\hat{Q}_2$. We then multiply $\hat{Q}_2$ by T_3 to obtain T_4. As the sums of the rows of T_5 are not exactly equal to the 1966 totals in the adjacent column we continue the iteration by calculating new P ratios which

* **Stone and his colleagues commence the operation in a slightly different way but the method shown here, which is somewhat simpler, gives the same final result.**

we designate $\hat{P}_3$. Applying $\hat{P}_3$ to T_4 we obtain T_5 and adding the columns of T_5 we obtain the figures in Row 28 which are very close to the required totals in Row 29 though not exactly the same. We could by continuing the iterations bring the totals even closer together but there is no need to do this. The results are sufficiently good for all practical purposes.

In practical work the research worker or planner has to go through all the steps outlined above in order to obtain the desired results but an account of the procedure can be summarized briefly by saying that in order to obtain a matrix of intermediate transactions for 1966 from a similar 1960 matrix designated T_o we postmultiply T_o by $\hat{Q}$ and then premultiply $T_o\hat{Q}$ by $\hat{P}$ to obtain $\hat{P}T_o$ where $\hat{P}$ and $\hat{Q}$ are diagonal matrices, i.e. matrices having zeros in all elements other than the principal diagonals. In this case

$$\hat{P} = \hat{P}_1\hat{P}_2\hat{P}_3 \text{ and } \hat{Q} = \hat{Q}_1\hat{Q}_2$$

Written out in full

$$\hat{P} = \begin{bmatrix} 1{\cdot}30301 & 0 & 0 \\ 0 & 1{\cdot}89480 & 0 \\ 0 & 0 & 2{\cdot}12279 \end{bmatrix} \begin{bmatrix} 0{\cdot}969554 & 0 & 0 \\ 0 & 1{\cdot}013624 & 0 \\ 0 & 0 & 0{\cdot}997212 \end{bmatrix}$$

$$\begin{bmatrix} 0{\cdot}997776 & 0 & 0 \\ 0 & 1{\cdot}000213 & 0 \\ 0 & 0 & 1{\cdot}001397 \end{bmatrix} = \begin{bmatrix} 1{\cdot}260529 & 0 & 0 \\ 0 & 1{\cdot}921024 & 0 \\ 0 & 0 & 2{\cdot}119829 \end{bmatrix}$$

$$\hat{Q} = \begin{bmatrix} 0{\cdot}662918 & 0 & 0 \\ 0 & 1{\cdot}039805 & 0 \\ 0 & 0 & 1{\cdot}133765 \end{bmatrix} \begin{bmatrix} 0{\cdot}992893 & 0 & 0 \\ 0 & 1{\cdot}002504 & 0 \\ 0 & 0 & 0{\cdot}994579 \end{bmatrix}$$

$$= \begin{bmatrix} 0{\cdot}658207 & 0 & 0 \\ 0 & 1{\cdot}042409 & 0 \\ 0 & 0 & 1{\cdot}127619 \end{bmatrix}$$

To check that our calculations are correct we multiply the 1960 matrix of intermediate transactions (T_o) by $\hat{P}$ and $\hat{Q}$ as follows:

$$\hat{P}T_o = \begin{bmatrix} 1{\cdot}260529 & 0 & 0 \\ 0 & 1{\cdot}921024 & 0 \\ 0 & 0 & 2{\cdot}119829 \end{bmatrix} \begin{bmatrix} 2{\cdot}180 & 81{\cdot}687 & 1{\cdot}143 \\ 27{\cdot}709 & 98{\cdot}036 & 25{\cdot}457 \\ 11{\cdot}020 & 32{\cdot}242 & 19{\cdot}487 \end{bmatrix}$$

$$= \begin{bmatrix} 2{\cdot}74795 & 102{\cdot}96883 & 1{\cdot}44079 \\ 53{\cdot}22965 & 188{\cdot}32951 & 48{\cdot}90351 \\ 23{\cdot}36052 & 68{\cdot}34753 & 41{\cdot}30911 \end{bmatrix}$$

$$\hat{P}T_0\hat{Q} = \begin{bmatrix} 2{\cdot}74795 & 102{\cdot}96883 & 1{\cdot}44079 \\ 53{\cdot}22965 & 188{\cdot}32951 & 48{\cdot}90351 \\ 23{\cdot}36052 & 68{\cdot}34753 & 41{\cdot}30911 \end{bmatrix} \begin{bmatrix} 0{\cdot}658207 & 0 & 0 \\ 0 & 1{\cdot}042409 & 0 \\ 0 & 0 & 1{\cdot}127619 \end{bmatrix}$$

$$= \begin{bmatrix} 1{\cdot}80872 & 107{\cdot}33564 & 1{\cdot}62466 \\ 35{\cdot}03613 & 196{\cdot}31638 & 55{\cdot}14453 \\ 15{\cdot}37606 & 71{\cdot}24608 & 46{\cdot}58094 \end{bmatrix} \qquad (6.1)$$

It will be noted that except for rounding errors, these figures are the same as those in the last section of Table 6.2. Hence our calculations are correct. In doing these checks we must be careful to do the multiplications in the proper order otherwise we do not get the correct result.

After calculating and checking the intermediate transactions for 1966, the next operation is to calculate technical coefficients for that year by dividing the columns of figures in the last section of Table 6.2 by the corresponding 1966 total outputs given in line 30 of Table 6.2 also.

We set down the coefficients obtained by this exercise in Table 6.3, together with the corresponding technical coefficients for 1960, as shown in Table 6.3.

TABLE 6.3 Intermediate Technical Coefficients for 1966 and 1960

	1966 Coefficients			1960 Coefficients		
	Agric.	Industry	Services	Agric.	Industry	Services
		$_1A$			$_0A$	
Agric.	0·0073	0·1147	0·0028	0·0109	0·1518	0·0038
Industry	0·1405	0·2099	0·0957	0·1383	0·1822	0·0845
Services	0·0617	0·0762	0·0808	0·0550	0·0599	0·0647

After setting out the intermediate coefficients for the two years, the next exercise is to obtain the relationships between the corresponding individual coefficients in both years for the purpose of projecting the 1966 figures forward to 1972. Since 1966 is halfway between 1960 and 1972 it is assumed that the changes between 1960 and 1966 will apply again between 1966 and 1972. If of course we think that this assumption is too naive we can always, as shown later, single out important coefficients for special treatment. For the purpose of the present exercise, however, we treat all coefficients in the same way.

The simplest method of finding the relationships between the 1966 and 1960 figures would be to divide the 1966 coefficients by the corresponding ones for 1960 to obtain a matrix of ratios between the individual figures in the two years. This method, however, is not favoured by the experts as it is considered that coefficients do not change in such a simple way. More subtle

changes occur over time as a result of changing technology and scale of operations.

The method suggested by Stone and his associates is to find two diagonal matrices

$$\begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix}$$

written $\hat{R}$ and $\hat{S}$ for short, such that the matrix of intermediate technical coefficients for 1966 called $({}_1A)$ is given by premultiplying the corresponding matrix for 1960 $({}_oA)$ by $\hat{R}$ to obtain $\hat{R}({}_oA)$, and postmultiplying $\hat{R}\,({}_oA)$ by $\hat{S}$ to obtain $\hat{R}({}_oA)\hat{S}$. Hence the method is referred to as the "RAS" method. Written algebraically:

$$({}_1A) = \hat{R}({}_oA)\hat{S} \tag{6.2}$$

If therefore the intermediate technical coefficients for 1960 and 1966 in symbols are:

$$({}_oA) = \begin{bmatrix} {}_oa_{11} & {}_oa_{12} & {}_oa_{13} \\ {}_oa_{21} & {}_oa_{22} & {}_oa_{23} \\ {}_oa_{31} & {}_oa_{32} & {}_oa_{33} \end{bmatrix} : ({}_1A) = \begin{bmatrix} {}_1a_{11} & {}_1a_{12} & {}_1a_{13} \\ {}_1a_{21} & {}_1a_{22} & {}_1a_{23} \\ {}_1a_{31} & {}_1a_{32} & {}_1a_{33} \end{bmatrix} :$$

the relationship between the old and new coefficients can be expressed as follows:

$$\begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{bmatrix} \begin{bmatrix} {}_oa_{11} & {}_oa_{12} & {}_oa_{13} \\ {}_oa_{21} & {}_oa_{22} & {}_oa_{23} \\ {}_oa_{31} & {}_oa_{32} & {}_oa_{33} \end{bmatrix} \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix} = \begin{bmatrix} {}_1a_{11} & {}_1a_{12} & {}_1a_{13} \\ {}_1a_{21} & {}_1a_{22} & {}_1a_{23} \\ {}_1a_{31} & {}_1a_{32} & {}_1a_{33} \end{bmatrix} \tag{6.3}$$

or alternatively

$$\begin{bmatrix} r_1({}_oa_{11})s_1 & r_1({}_oa_{12})s_2 & r_1({}_oa_{13})s_3 \\ r_2({}_oa_{21})s_1 & r_2({}_oa_{22})s_2 & r_2({}_oa_{23})s_3 \\ r_3({}_oa_{31})s_1 & r_3({}_oa_{32})s_2 & r_3({}_oa_{33})s_3 \end{bmatrix} = \begin{bmatrix} {}_1a_{11} & {}_1a_{12} & {}_1a_{13} \\ {}_1a_{21} & {}_1a_{22} & {}_1a_{23} \\ {}_1a_{31} & {}_1a_{32} & {}_1a_{33} \end{bmatrix} \tag{6.4}$$

As the matrices in (6.4) are equal, element by element

$$r_1({}_oa_{11})s_1 = {}_1a_{11};\ r_1({}_oa_{12})s_2 = {}_1a_{12};\ r_1({}_oa_{13})s_3 = {}_1a_{13}$$

and so on for the other entries.

It will be noticed that each row of the matrix on the left-hand side of (6.4) has a common r factor and each column has a common s factor. The r factors

are called the *substitution* factors, i.e. the factors that adjust each column for substitution effects. Since a different r is applied to each coefficient in a column, the r's bring about changes in the proportions in which the different inputs are used. For example if $r_1 = 0{\cdot}5, r_2 = 2{\cdot}0$ and the s value concerned is $1{\cdot}0$, the proportion in which input (1) is used in the final year is only half what it was in the base year, while the proportion of input (2) is doubled. The r's will usually also change the proportions in which intermediate and primary inputs are used in the base and final year but it could happen (though it is rather unlikely) that when the r's are applied to all the intermediate coefficients in a column, any increases in some coefficients might be counterbalanced by decreases in others, leaving the primary input coefficients virtually unchanged.

The s factors are called the *fabrication* factors because they always change the proportions in which intermediate and primary inputs (also known as the factors or fabricants of production) are used in the production of a commodity. For example if the s factor applied to any column is $0{\cdot}5$ its application (if we ignore the r effect) will result in halving the proportion of total intermediate inputs in the column. This means that the total of the primary input coefficients will have to be increased in order to bring the sum of all the coefficients in the column to unity.

Calculating $\hat{R}$ and $\hat{S}$

The system of equations in (6.3),

$$\text{i.e.}\quad \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{bmatrix} \begin{bmatrix} {}_0a_{11} & {}_0a_{12} & {}_0a_{13} \\ {}_0a_{21} & {}_0a_{22} & {}_0a_{23} \\ {}_0a_{31} & {}_0a_{32} & {}_0a_{33} \end{bmatrix} \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix} = \begin{bmatrix} {}_1a_{11} & {}_1a_{12} & {}_1a_{13} \\ {}_1a_{21} & {}_1a_{22} & {}_1a_{23} \\ {}_1a_{31} & {}_1a_{32} & {}_1a_{33} \end{bmatrix}$$

can be solved for the r's and s's. However a unique solution is not possible. This is easily demonstrated, for, by multiplying the elements of $\hat{R}$ by any constant scalar $k(\neq 0)$ and dividing the elements of $\hat{S}$ by k we leave ${}_0A$ and ${}_1A$ unchanged but vary the values which can be given to the r's and s's.

$$\begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{bmatrix} \begin{bmatrix} \\ {}_0A \\ \\ \end{bmatrix} \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix} = \begin{bmatrix} \\ {}_1A \\ \\ \end{bmatrix}$$

$$k \cdot \frac{1}{k} \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{bmatrix} \begin{bmatrix} \\ {}_0A \\ \\ \end{bmatrix} \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix} = \begin{bmatrix} \\ {}_1A \\ \\ \end{bmatrix}$$

$$k \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{bmatrix} \begin{bmatrix} \\ {}_0A \\ \\ \end{bmatrix} \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix} \frac{1}{k} = \begin{bmatrix} \\ {}_1A \\ \\ \end{bmatrix}$$

$$\begin{bmatrix} r_1k & 0 & 0 \\ 0 & r_2k & 0 \\ 0 & 0 & r_3k \end{bmatrix} \begin{bmatrix} \\ {}_0A \\ \\ \end{bmatrix} \begin{bmatrix} s_1/k & 0 & 0 \\ 0 & s_2/k & 0 \\ 0 & 0 & s_3/k \end{bmatrix} = \begin{bmatrix} \\ {}_1A \\ \\ \end{bmatrix}$$

Thus (kr_i) and $(\frac{1}{k}s_j)$, used as substitution and fabrication factors, give $r_i\,{}_0a_{ij}\,s_j$ for the typical element of ${}_1A$.

While a unique solution is not possible, two points emerge from the above.

(a) The ratios of the r's to each other are unique. If a set of r's is r_1', r_2', r_3' then

$$r_1 : r_2 : r_3 = r_1' : r_2' : r_3'$$

which indicates that

$$\frac{r_1}{r_1'} = \frac{r_2}{r_2'} = \frac{r_3}{r_3'} = k \text{ (where } k \text{ is any constant} \neq 0)$$

Similarly for the s's.

(b) The products r_is_j are constant (for all i, j) irrespective of the values assigned to the r's and s's as long as (a) is satisfied i.e. $r_is_j = r_i' s_j'$ and it is only the product that is of interest for each a_{ij}. Thus by assigning a value to one of the r_i and s_j variables one can obtain absolute values for all the other variables in the system. We proceed as follows.

First we substitute the 1960 and 1966 technical coefficients from Table 6.3 for the a's in (6.4) and obtain the system of equations in (6.5):

$$\begin{bmatrix} r_1(0{\cdot}0109)s_1 & r_1(0{\cdot}1518)s_2 & r_1(0{\cdot}0038)s_3 \\ r_2(0{\cdot}1383)s_1 & r_2(0{\cdot}1822)s_2 & r_2(0{\cdot}0845)s_3 \\ r_3(0{\cdot}0550)s_1 & r_3(0{\cdot}0599)s_2 & r_3(0{\cdot}0647)s_3 \end{bmatrix} = \begin{bmatrix} 0{\cdot}0073 & 0{\cdot}1147 & 0{\cdot}0028 \\ 0{\cdot}1405 & 0{\cdot}2099 & 0{\cdot}0957 \\ 0{\cdot}0617 & 0{\cdot}0762 & 0{\cdot}0808 \end{bmatrix} \quad (6.5)$$

From this system, by selecting the more substantial coefficients, and taking $r_1 = 1{\cdot}0$,

$$r_1s_2 = \frac{0{\cdot}1147}{0{\cdot}1518}, \quad \text{giving } s_2 = 0{\cdot}75560 \quad (6.6)$$

$$r_2s_2 = \frac{0{\cdot}2099}{0{\cdot}1822} = 0{\cdot}75560\ r_2 = 1{\cdot}152031$$

$$r_2 = 1{\cdot}52466 \quad (6.7)$$

$$r_2 s_1 = \frac{0{\cdot}1405}{0{\cdot}1383} = 1{\cdot}52466 \quad s_1 = 1{\cdot}015907$$

$$s_1 = 0{\cdot}66632 \qquad (6.8)$$

$$r_2 s_3 = \frac{0{\cdot}0957}{0{\cdot}0845} = 1{\cdot}52466 \quad s_3 = 1{\cdot}132544$$

$$s_3 = 0{\cdot}74282 \qquad (6.9)$$

$$r_3 s_3 = \frac{0{\cdot}0808}{0{\cdot}0647} = 0{\cdot}74282 \quad r_3 = 1{\cdot}248841$$

$$r_3 = 1{\cdot}68122 \qquad (6.10)$$

In practice one should use the numerically largest pairs of technical coefficients available (i.e. ${}_0a_{ij}$ *and* ${}_1a_{ij}$) and to minimize rounding errors each coefficient used should have at least six decimal places. It is necessary to check these results by substitution into the remaining equations so as to ensure that the r's and s's obtained from one set of equations in the system are the same as those which would be obtained if another set were used. If different results are obtained by using different sets of equations a mistake has been made somewhere which will have to be located. In this example checks made show that the above results are correct. We therefore set out the r and s values in matrix form below.

$$\hat{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1{\cdot}52466 & 0 \\ 0 & 0 & 1{\cdot}68122 \end{bmatrix} \quad \text{and } \hat{S} = \begin{bmatrix} 0{\cdot}66632 & 0 & 0 \\ 0 & 0{\cdot}75560 & 0 \\ 0 & 0 & 0{\cdot}74282 \end{bmatrix} \qquad (6.11)$$

It should be noted that if we had assigned a different value for r_1 in doing our calculations, different values for all the variables would be obtained. This would not affect our results, however, because if we reduced the r values we would increase the s values by a corresponding fraction to give the same result when a particular r and s is applied to a coefficient. Thus in the above example if we assigned a value of 2·0 to r_1, s_1 would become $\frac{0{\cdot}66632}{2} = 0{\cdot}33316$ and similarly for other r and s values.

Projecting Forward the 1966 Coefficients

Technical coefficients for 1972 are estimated by multiplying the 1966 coefficients given in Table 6.3 by the r's and s's given in (6.11).

Thus:

$$A_{72} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1{\cdot}52466 & 0 \\ 0 & 0 & 1{\cdot}69122 \end{bmatrix} \begin{bmatrix} 0{\cdot}0073 & 0{\cdot}1147 & 0{\cdot}0028 \\ 0{\cdot}1405 & 0{\cdot}2099 & 0{\cdot}0957 \\ 0{\cdot}0617 & 0{\cdot}0762 & 0{\cdot}0808 \end{bmatrix}$$

$$\begin{bmatrix} 0{\cdot}66632 & 0 & 0 \\ 0 & 0{\cdot}75560 & 0 \\ 0 & 0 & 0{\cdot}74282 \end{bmatrix} = \begin{bmatrix} {\cdot}00486 & {\cdot}08667 & {\cdot}00208 \\ {\cdot}14274 & {\cdot}24181 & {\cdot}10838 \\ {\cdot}06912 & {\cdot}09680 & {\cdot}10091 \end{bmatrix} \qquad (6.12)$$

It should be kept in mind that the above projections for 1972 relate to the intermediate coefficients. Coefficients for primary inputs have to be projected separately, subject to the constraint that the total of the latter in any column plus the intermediate coefficients in the same column add to unity. The sum of the intermediate coefficients in the first column of (6.12) is 0·2167, hence the total of the primary input coefficients for this column is 1·0000 — 0·2167 = 0·7833. Similarly it can be determined that the total of the primary input coefficients for the second and third columns are 0·5747 and 0·7886 respectively. The totals of the intermediate and primary input coefficients in 1960 and 1972 are compared in Table 6.4.

TABLE 6.4 Comparison of Total Intermediate and Primary Input Coefficients in 1960 with the 1972 projections

	Agriculture		Industry		Services	
	1960	**1972**	**1960**	**1972**	**1960**	**1972**
Intermediate Inputs	0·2042	0·2167	0·3939	0·4253	0·1530	0·2114
Primary Inputs	0·7958	0·7833	0·6061	0·5747	0·8470	0·7886
Total	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000

It is difficult to say if the projected coefficients are realistic, and this is particularly so for a highly aggregated model like this. For a less highly aggregated model the research worker may have some ideas about the likely magnitudes of certain coefficients in future years so that he has some standard of comparison, however crude. In practice of course a good independent estimate of a coefficient should always be used in preference to an RAS projection which could be in considerable error. Indeed if information of a technical or other nature is available, it should always be used to the fullest extent in making projections, and the RAS method only used to project coefficients about which little information is available. For example, in calculating the entries in Table 6.2 it was assumed that no information was available about any of

the intermediate transactions for 1966. In practice many of the 1966 transactions would be known in 1968 and these should therefore be omitted from the calculations. This is done by putting zeros in the cells of the 1960 table for which known entries are available in 1966 and working on the remaining cells. If this is done the magnitudes of the known entries are also omitted from the intermediate row and column totals so as to obtain correct $\hat{P}$ and $\hat{Q}$ ratios. An example will explain clearly how this is done.

Suppose that in 1968 we had information as to the sales of agricultural produce to industry in 1966 and of industrial produce to agriculture (i.e. the figures for 1966 corresponding to 81·687 and 27·709 in the first section in Table 6.2). Suppose for the sake of argument that these figures for 1966 were £104m and £40m respectively. The table corresponding to the first section of 6.2 for this situation would therefore appear as Table 6.5 and the iterations would proceed from this base in exactly the same way as in Table 6.2. The two blank cells will, however, remain blank throughout the iterations. When the iterations are completed the known figures of £104m and £40m can be entered in the blank cells, but if this is done we cannot calculate unique sets of $\hat{R}$ and $\hat{S}$ values from the resulting technical coefficients.

TABLE 6.5 Transactions Table for 1960 with Two Blank Cells and Adjusted Totals for 1960 and 1966

	Intermediate Transactions 1960				Total Intermediate 1966 (b)	Ratio (b)/(a)
	Agric.	Industry	Services	Total (a)		
Agriculture	2·180	-	1·143	3·323	6·769	2·0370
Industry	-	98·036	25·457	123·493	246·497	1·9960
Services	11·020	32·242	19·487	62·749	133·203	2·1228
Total 1960	13·200	130·278	46·087	189·565	386·469	
Total 1966	12·196	270·996	103·277			

We have mentioned above that the intermediate coefficients calculated for 1966 are from the nature of their derivation not independent either of one another or of the corresponding 1960 figures. Because of this it is possible to calculate unique $\hat{R}$ and $\hat{S}$ operators* which link the 1960 and 1966 figures. If however we were to include zeros in some of the 1960 cells and later replace these zeros by independent figures from outside the system, we could not then derive sets of $\hat{R}$'s and $\hat{S}$'s which would fit all the entries. Hence it must be men-

* Unique in the sense that $r_i s_j$ is invariant.

tioned that if any cell is left blank in deriving the mid-year figures (i.e. 1966 in this case), this cell will also have to be left blank in making the forward projections. We cannot project forward to 1972 by the RAS method a 1966 figure determined independently of the system. Independent projections have therefore to be made for blank cells and the results integrated into the matrix by proportional adjustment of the RAS-derived coefficients in the relevant columns so that the column totals are unity in each case.

Projections of individual coefficients can be made in various ways depending on the coefficients concerned. In some cases a simple linear extrapolation of past trends in a coefficient may be best. In other cases regression methods of projection may be used, but no matter what method is adopted the forecaster should study closely the industries concerned so as to obtain ideas of the technical and demand changes which are likely to occur. This may help considerably in making the projections, but at worst it will prevent the forecaster from obtaining ridiculous results.

The Use of RAS for Updating Input–Output Models

A serious disadvantage associated with input–output analysis is that the tables are out of date by the time they are constructed. This is more or less inevitable as much of the required basic data become available very slowly. Especially is this true for Census of Production figures, which may be several years in arrears by the time they are finally published.

On the other hand some basic data become available fairly quickly. As was shown above it was possible to fill in the borders* of a 1966 table in 1968 and some large inter-industry flows might be known. In addition to having the borders of the table available fairly quickly a number of intersectoral flows may become available at a short interval also. For example, all the row entries for the Irish agricultural sector in any year are available within about three months of the end of the year. Many, though not all, of the column entries are available at this time also. A considerable number of industrial entries are also available early the following year, so that by using the available data and the RAS method it should be possible to have an input–output table for any one year towards the end of the following year. This table would of course be fairly approximate, but its timeliness would outweigh this disadvantage. A table such as this would also be cheap to construct, which is a further advantage.

There is therefore a strong case for using the RAS method to construct annual input–output tables which can be revised as more information comes to hand. It should not be necessary to revise each year's table and indeed if

* Borders in this context represent the entries surrounding the matrix of unknown intermediate demands, i.e. final demands, primary inputs, and total outputs of the different sectors are the borders.

a full revision were made every four or five years, adequate tables on an annual basis could be prepared as a routine procedure for the intervening years. This use of the RAS method is therefore probably more important than its use as a forecaster of technical coefficients, though of course its use for the latter purpose should not be discounted.

Other Methods of Updating Input–Output Models

There are many possible methods of updating transactions tables, apart from the RAS method described above. Two methods, namely the least-squares method and the time-series method, might be mentioned as examples of ways other than the RAS way of updating inter-industry structures.

One author has a paper [1] on the least-squares method, which is closely related to that of the RAS. It is based on the mathematical technique known as constrained least squares with Lagrange multipliers. It updates the inter-industry structure to fit specified totals for rows and columns of inter-industry transactions and provides row and column multipliers to be used for projections. In order to calculate these multipliers it is necessary to solve a large number of simultaneous linear equations and this operation would require the use of a computer. Thus the RAS method has the advantage of being applicable via desk calculators without the need for computers although of course computer processing is advisable for applying RAS to large tables.

The time-series method involves a study of each important coefficient, as a mathematical function of time. For a simple description or model, we might suppose that an observed approximate average change of x per cent per year, starting with a specified base year, has been recorded or derived for a certain coefficient. Then this rate of change, x per cent per annum, may be used to predict that coefficient for a specified future year. In order to derive the mathematical function of time for the rate of change of a coefficient, it is advisable to have at least six observations and to use regression or other methods of fitting the function to the data.

REFERENCE - CHAPTER 6

[1] Henry, E.W., "Relative efficiency of RAS versus Least Squares methods of updating input–output structures", *Economic and Social Review,* Vol. 5, No.1 and No. 2, Dublin (Oct. 1973 and Jan. 1974).

Section 3

APPLICATIONS OF THE INPUT–OUTPUT SYSTEM

7 USING PARTIAL TRANSACTIONS TABLES TO EXAMINE MILK AND BEEF POLICIES

The input–output system has been widely used in economic analysis of all kinds, and new forms of both theory and application are being tried out and developed year by year. New ideas and applications are usually offered for discussion at World Input–Output Conferences organized at three-yearly intervals by Leontief and his associates of the Havard Economic Research Project. The papers read at the input–output conferences are published subsequently in Proceedings issues, and interested readers are referred to such issues for a comprehensive picture of the field.[1] It is true that many of the applications described in the Proceedings are presented in highly technical language and are best understood by relatively specialized methematicians. Non-mathematical economists have therefore to look elsewhere for an insight into what is happening in the input–output field. The authors' sympathies lie with the latter people and for that reason we have given in this and subsequent chapters a few applications of input–output in as simple a language as possible. We should hasten to add that the technique does not lend itself very readily to simple exposition and students may find some of the examples not as easy to follow as they would wish. The application given in this chapter, however, should be readily understood by readers who have persevered with our exposition so far.

The Problem and its Background

Though its importance is decreasing over the years, agriculture is still one of the largest sectors in the Irish economy, and within agriculture, milk and beef production are by far the most important enterprises. Because the home population is low (about 3 million people) the country consumes only about one-third of total production of these commodities, the balance being exported.

Prior to Ireland's decision to join the European Economic Community it was difficult to find good export markets for agricultural produce. The situation was not too bad for beef but dairy produce had encountered real problems over the years. The free trade area agreement with Britain in 1966 eased this situation somewhat by providing a guaranteed outlet at fairly reasonable prices for about three-quarters of Ireland's dairy produce exports. Unfortunately, however, produce sent to other countries usually had to be sold at very low prices.

The poor export prices for dairy produce placed a considerable strain on the

Irish exchequer and was a matter of serious concern for the government. As policy was organized, farmers were paid a guaranteed price for all milk sold. Except for a small subsidy on butter, economic prices were charged for dairy produce sold on the home market, but the exchequer had to pay for the loss incurred on export markets. By 1968 despite the Anglo-Irish Free Trade Agreement this loss had become serious by Irish standards. In that year the equivalent of 49·5 million gallons of milk was exported to markets other than Britain at an average price of about 2·7 p per gallon, compared with a price of about 10·7 p per gallon received by farmers for the same milk. Export subsidy payments on the latter were therefore about £4 million and this was only part of the total export subsidy bill, milk products sold on the British market also having had to be subsidized, though at a lower rate for a much larger quantity.

Because the situation looked like deteriorating further in future years the government sought suggestions from many quarters as to remedies. The most obvious solution was to replace the low-priced milk by beef from suckled cows, but if such substitution were to take place the returns to farmers from the beef cows would need to be kept in line with those from dairying. Unfortunately the beef would also require some element of subsidization, despite the apparently good market available. In this climate of opinion one of the authors was asked to analyse the situation, suggest alternatives to dairying, and estimate the subsidy levels which might entice some farmers to go into other lines of production. The method of carrying out the analysis, the results of which are described elsewhere,[2] is discussed below.

In the original paper a number of farming systems were compared with dairying, but as the method of analysis used was the same for all systems we will confine our attention here to a comparison of dairying with single suckling. Under the latter system all of the cow's milk is suckled by one calf and, after weaning, the calf is fed fairly well and sold fat at 2 years of age. Under the dairying system all the milk produced (except for a small amount fed to calves) is sold for manufacturing purposes while the calves other than those required for replacement are sold for beef at 2½ to 3 years of age.

In making the analysis, the systems under review were compared by using sectoral input–output transactions tables. As a first step an assessment was made of the resources used in the production of the 49·5 million gallons of low-priced milk and of the beef cattle produced along with it as a joint product (System (1)). The benefit to the economy generally from this production was also estimated by using 1968 yields, prices, and subsidy levels. The results of this assessment are shown in Table 7.1.

SYSTEM (1) DAIRYING AND CATTLE

It was estimated that the amount of very low-priced milk exported in 1968

(49·5m gallons) would be produced by 1000 000 cows and that these cows and their followers, i.e. heifers for replacement, and dry cattle for slaughter, would require about 529 000 acres of average land. The financial flows for this system as set out in Table 7.1 are described below, commencing with the dairying sector.

Dairying Sector

The output of the dairying sector includes, in addition to milk sold, the value of calves other than dairying replacements transferred to the cattle herd, as well as milk, whole and skim, fed to these calves. The imputed value of these transfers is deducted from cattle sales in obtaining income from the latter sector. After allowing for mortality of all cattle, cows and calves, it was estimated that 100 000 cows produce 85 000 cattle, either for replacement of the dairy herd or for sale as beef. Of these surviving cattle 69 000 are sold as fresh beef at the age of 2½ to 3 years and 16 000 replace a similar number of cows and bulls which are sold for dead meat also. The calves which ultimately go for dead meat are sold by the dairying sector to the cattle sector at an average price of £15 each. No value is placed on the calves retained in the dairying sector for replacements. These are netted out of the system.

The 100 000 dairy cows produce 53 million gallons of milk of which 3·5 million gallons are used for feeding calves and the remaining 49·5 million gallons are sold to the milk processing sector at 10·7 p per gallon. Of the whole milk fed to calves 0·7 million gallons are used for replacement calves and are not valued. The remaining 2·8 million gallons are used for the beef calves and are valued at the sale price at 10·7 p per gallon. This milk is assumed to be sold from the dairying sector to the cattle sector.

Milk processing returns 8·6 million gallons of skim milk to the dairying sector and retains the balance of some 40 million gallons. It pays the dairying sector for 31 million gallons at the rate of 1·1 p per gallon, and in accordance with traditional usage keeps the remainder free. Of the skim milk returned from the creamery, 1·8 million gallons are fed to replacement calves and are not valued, while the remaining 6·8 million gallons are sold to the cattle sector at the creamery price of 1·1 p per gallon.

The output value of the dairy herd is given in the Dairying row of Table 7.1. The figure in (£000) of 1437 in the Cattle column of this row represents the value of calves and of whole and skim milk sold by dairying to the cattle sector. The figure of 5652 in the Milk Processing column is the value of whole and skim milk sold to creameries, while the figure of 894 in the Animal Slaughter column is the value of old cows and bulls sold to meat factories. The prices used for the latter are the average prices paid by factories for such animals. All prices were supplied by the Central Statistics Office.

Inputs to the dairying sector are given in the Dairying column of the table.

TABLE 7.1 System (1) Input–Output Table for 100 000 Dairy Cows in 1968 (present situation) (£000)

Outputs → / Inputs ↓	Inter-Industry									Final Demand			
	Dairying	Cattle	Barley	Silage	Pasture	Grain Milling	Milk Processing	Animal Slaughter	Other Intermediate	Other Industries	Exports	Output	F.O.B. Value of Exports
Dairying		1 437					5 652	894				7 983	
Cattle								5 651	3			5 654	
Barley						830						830	
Silage	602	547										1 149	
Pasture	296	325										621	
Grain Milling	512	532										1 044	
Milk Processing											5 511	5 511	2 157
Animal Slaughter									931		6 405	7 336	6 170
Other Intermediate						21				630	838	1 489	838
Primary Inputs	837	524	396	1 241	744	133	1 241	731	287			6 134	
Subsidies			−38	−92	−123		−1 796	−484			−2 233*	−4 766	
Income Arising	5 736	2 289	472			60	414	544	268			9 783	
Total Inputs	7 983	5 654	830	1 149	621	1 044	5 511	7 336	1 489	630	10 521	42 768	9 165
Acres (000)	248	257	24									529	
Income per Acre (£)	23·1	8·9	19·4										

* In addition to the state export subsidy of £2·2 million, an additional subsidy of half this amount was required to export the milk in question. This subsidy was recouped by way of a levy on producers and therefore represents a redistribution of income within the national dairying sector.

The figures 602 and 296 for Silage and Pasture respectively are estimated amounts of these items consumed by cows and replacement calves, valued at cost-of-production prices. The figure of 512 in the Grain Milling row is the estimated value of meals (at grain millers' ex-store prices) consumed by dairy cows and replacements. Distribution margins on these meals are included in other primary inputs in the dairying column. Also included in primary inputs in the dairying column are rates on buildings, veterinary fees and medicines, transport and marketing costs, etc. Subsidies are not included in the Dairying column as they are not paid directly to farmers. Certain production-type subsidies paid to the Milk Marketing Board are included in the Milk Processing column while the direct export subsidy is included in the Export column of the Subsidy row. The figure of 5736, for income arising in the Dairying column, is the difference between total output and all inputs mentioned above.

The entry of 248 in the Acres row of the Dairying column is the acreage of silage and pasture required by the dairy cows and their followers. It is estimated that these animals require about 162 000 acres of pasture and 86 000 acres of silage, the latter acreage producing some grazing as well. The income of £23·1 per acre is obtained by dividing the income arising by the acreage of silage and pasture (5736/248 = 23·1).

Cattle Sector

The entry of 5651 in the Animal Slaughter column of the Cattle row is the value of 69 000 fat cattle sold to animal slaughter, while the entry of 3 in the Other Intermediate Industries' column is the value of casualty hides of all animals sold to the fellmongery and tanning industry. Strictly speaking the casualty hides of cows and bulls should have been entered in the Dairying row of the Other Industries column but their entry in the Cattle row makes no significant difference to the results.

The inputs to the cattle sector as given in the Cattle column are calves plus whole and skim milk purchased from dairying (1437); silage (547), and pasture (325) valued at cost-of-production prices, and meals fed to cattle (532) valued at grain millers' ex-store prices. The entry of 524 for Primary Inputs includes the same items as for dairying, while the entry of 2289 for Income Arising is the difference between total output and all the above inputs. It is estimated that cattle require (in thousands of acres) 178 of pasture, 79 of silage. When the income arising is divided by the sum of these figures we obtain the income per acre, i.e. 2289/257 = 8·9.

Barley Sector

It is assumed that all barley is sold off farms and later repurchased by dairy farmers and cattle producers in the form of barley meal and compound feeds. Hence the barley acreage is not attributable to the livestock sectors. The total

acreage of barley is 24 000. All produce except seed is assumed sold to grain milling. Average 1968 yield and output prices are used in obtaining the value of output. Seed is omitted from output and inputs, but since practically all barley seed is sold off farms, distribution charges on seed are included as a cost.

Primary inputs of the barley sector include the latter cost together with land annuities, rates of land, fertilisers, and other expenses based on data from various sources, mainly Agricultural Institute figures. The entries for land annuities and fertilisers include subsidies which are deducted in the subsidy row. Income arising per acre of barley is estimated at £19·4 per acre.

Silage and Pasture Sectors

It is assumed that each acre of silage yields about 27 cwt of barley equivalent, about 1 ton of this being silage and the remainder grazing. It is assumed that pasture yields just over one ton of barley equivalent per acre. All pasture and silage are used by the cattle and dairying sectors, and the pasture and silage sectors have zero profits. Primary inputs to these sectors consist of the same items as for barley, while subsidies are those on land annuities and fertilisers as in the case of barley.

Grain Milling Sector

The output value of 1044 in the Grain Milling row is the value of meals sold by grain milling to cattle and dairying. Inputs to the grain milling sector are the value of the barley purchased from the barley sector, some minerals and supplements purchased from other industries and primary inputs, which include some imported soya-bean meal. The magnitude of the primary inputs, other than imports, and income arising in this and all the other non-farming intermediate sectors, are based on the technical coefficients in the O'Connor–Breslin 1964 agricultural input–output model. [3]

Milk Processing Sector

The output of milk processing (5511) is the estimated value of butter and skim milk powder exported to markets other than Britain and Northern Ireland at the prices which the creameries received for these items from the Milk Marketing Board. In the Milk Processing row is also given the f.o.b. export value of these items (2157) for comparative purposes.

The inputs of milk processing are the values of the milk, whole and skim, purchased from dairying together with primary inputs and subsidies.

Animal Slaughter Sector

The output of the animal slaughter sector (7336) is made up of exports of dead meat (6405), and hides and offals sold to other intermediate industries (931). These are valued at prices received by the factories, which include

subsidies. The subsidies of the dead meat, estimated at 484, are entered in the Subsidy row of the Animal Slaughter column.

Other Intermediate Industries

This sector is a group of industries including fellmongery, tanning, fats and oils, which purchase hides, skins, fats and offals from animal slaughter, and sell products to exports and to other industries (mainly leather footwear) not included in the intermediate sector of the table. All the entries in the row and column of this sector are based on the 1964 input–output coefficients referred to above.

SYSTEM (2) SINGLE SUCKLING

After constructing the partial transactions table for System (1), the next exercise was to construct a similar table for the alternative system – single suckling. Details of this system are given in Table 7.2.

In constructing Table 7.2 the same principles were adopted as for Table 7.1. The land area was kept constant at 529 000 acres. Yields of all crops were taken as being the same, but the crop proportions were changed due to changed feed requirements.

The number of cows carried under System (2) was estimated at 138 000. From these about 100 000 cattle were available for sale as fresh or chilled beef at the age of 2 years, in March and April. The remainder, other than mortality, were assumed to go for herd replacement. At slaughter, bullocks were assumed to weigh 10·1 cwt and heifers 8·8 cwt liveweight.

In making valuations, cull cows and bulls were valued at prices paid by factories for such animals in 1968. Corresponding prices were, however, not available for the single sucklers and we therefore had to work back from the f.o.b. export value of the meat, using as far as possible the 1964 input–output coefficients. These coefficients had, however, to be adjusted, particularly that for income arising in the animal slaughtering sector, since meat factory incomes are always rather low in the early part of the year compared with the annual average, to which the 1964 coefficients relate.

Comparison of the two Systems

The two systems, (1) dairying plus cattle and (2) single suckling, are compared in Table 7.3. As can be seen from this table the income arising in the farming sectors from the dairying and cattle system (System (1)) was £8·5 million of which £5·7 million came from dairying, £2·3 million from cattle and £0·5 million from barley used for feeding cows and cattle. The direct subsidies paid on this system were £4·5 million. Farm income less these subsidies was therefore £4·0 million. Income arising in the associated non-farming industries was £1·3 million, giving a total farm and non-farm income less sub-

TABLE 7.2 System (2) Input–Output Table for Single Suckling Cows (£000)

Outputs → / Inputs ↓	Inter-Industry							Final Demand			
	Cattle	Barley	Silage	Pasture	Grain Milling	Animal Slaughter	Other Inter-mediate	Other Indus-tries	Exports	Output	F.O.B. Value of Exports
Cattle						8 713	4			8 717	
Barley					1 595					1 595	
Silage	1 428									1 428	
Pasture	507									507	
Grain Milling	1 805									1 805	
Animal Slaughter							1 285		8 749	10 034	8 817
Other Intermediate								883	1 173	2 056	1 173
Primary Inputs	1 263	761	1 543	607	119	1 043	395		68	5 799	
Subsidies		– 73	–115	–100		–219				–507	
Income Arising	3 714	907			91	497*	372			5 581	
Total Inputs	8 717	1 595	1 428	507	1 805	10 034	2 056	883	9 990	37 015	9 990
Acres (000)	482	47								529	
Income per Acre (£)	7·7	19·4									

* Income arising in the animal slaughter sector is much lower than the corresponding figure in Table 7.1 because the single suckled cattle are assumed sold in March and April when cattle prices are high and factory profits are low.

TABLE 7.3 Comparison of Alternative Farming Systems

Category	Dairying and Cattle (1)	Single Suckling (2)
	£000	
Income Arising from:		
Dairying	5 736	–
Cattle	2 289	3 714
Barley	472	907
Total Farming Sectors	8 497	4 621
Direct Subsidies*	4 513	219
Farm Income less Subsidies	3 984	4 402
Subsidies required to maintain farm income at System (1) Level	4 513	4 095
Income Arising from:		
Grain Milling	60	91
Milk Processing	414	–
Animal Slaughter	544	497
Other Intermediate	268	372
Total Non-Farming Sectors	1 286	960
Total Farm and Non-Farm Income less subsidies	5 270	5 362
Product Exports (f.o.b. value)	9 165	9 990
Farm Income per acre of grassland from:		
Dairying	23·1 (6·9)	–
Cattle	8·9 (7·0)	7·7 (7·3)
Dairy and Cattle	15·9 (7·0)	–

* Subsidies paid on fertilisers for grass and barley production are omitted.
Note – Figures in brackets represent income per acre less subsidies.

sidies of £5·3 million. The f.o.b. value of the exports from this system was £9·2 million. The income arising per acre of grassland including subsidies was £8·9 from cattle, £23·1 from dairying, and £15·9 from dairying and cattle combined. When product subsidies were deducted the return per acre of grassland from dairying was £6·9, from cattle £7·0, and from dairying and cattle combined £7·0.

The income arising in the farming sector from single suckling on 529 000 acres was estimated at £4·6 million. Farm income less subsidies at £4·4 million was therefore only £0·4 million higher than the corresponding figure for System (1). Hence to keep farm income from single suckling as high as from dairying and cattle production would require a subsidy of £4·1 million which was little less than that required for System (1). Specialized beef production therefore was shown to be less profitable than many administrators had thought.

It was pointed out, however, in the original paper that from the national point of view the single suckling would be preferable to the low priced milk production. The various international projections indicated that the market outlook for milk was not as favourable as that for beef and for this reason the level of subsidies required to support dairying would be likely to increase while that required for beef was likely to diminish or at worst remain static.

An estimate was then made of the level of subsidy per cow which would be required in order to shift farmers from dairying to single suckling on the assumption that the 1968 level of dairying supports remained substantially unchanged. In doing this exercise the levels of farm incomes – including subsidies from dairying alone and from single suckling – were divided by the number of cows in each. Thus in System (1), income from dairying alone was found to be £57 per cow (5736 000/100 000), while that from single suckling was £27 (3 714 000/138 000) or a difference of about £30 per cow from the two enterprises.

It was pointed out that this difference must be interpreted carefully because dairying and single suckling are very unsimilar enterprises. As labour and capital requirements for the suckling are much less than those for dairying, a smaller subsidy than £30 per cow would be required to entice some farmers out of dairying. At the time of writing the original paper a small beef cow subsidy was in operation which was not included in the calculations. The subsidy seemed to have had little influence in increasing beef cows. Subsequent to the publication of this paper the beef cow subsidies were increased substantially, resulting in a considerable increase in beef cows in the state.

Conclusion

It is obvious from the above example that a simple sectoral input–output partial transactions table can be a powerful tool in comparing the effects of different types of subsidies on both the farm and non-farming sectors of an economy. Because sectoral tables are used, the full impact effects of systems examined in this way cannot be determined. Hence there is not much point in calculating technical and interdependence coefficients. For the purpose of the exercise under review, however, it was not necessary to do this.

Similar studies can also be carried out for industrial enterprises but one of the difficulties in this regard is the absence of good data. Leontief in his presidential address to the American Economic Association [4] has correctly pointed out that much better data are available for the agricultural sectors of the U.S. economy than for other sectors. The same is true in Ireland and indeed in all countries with which the authors are familiar. Hence if we are to make much progress with input–output studies of industrial enterprises better data than presently available will have to be provided.

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[1] See for example "Input–Output Techniques", *Proceedings of the Fifth International Conference on Input–Output techniques,* Geneva, January 1971. Eds. A. Brody and A.P. Carter, North Holland Publishing Co. (1972).

[2] O'Connor, R., "An Analysis of Recent Policies for Beef and Milk", *Journal of the Statistical and Social Inquiry Society of Ireland*, Vol. XXIII, Part II, 1969–70 issue.

[3] O'Connor, R., with Breslin, M., 'An Input–Output Model of the Agricultural Sector of the Irish Economy in 1964', Paper No. 45 of the *Economic and Social Research Institute,* Dublin (1968).

[4] Leontief, W., "Theoretical Assumptions and Nonobserved Facts", *The American Economic Review,* Vol. LXI, No. 1 (March 1971).

8 USING INPUT–OUTPUT TO EXAMINE AN IMPORT SUBSTITUTION PROBLEM

Introduction

Home grown cereal production has been supported by the Irish Government since the early 1930's, both by guaranteed prices and by import controls. This policy has resulted in relatively high food and feed prices compared with those in many other countries, and though high prices have been fixed for animal products to compensate for the high feed costs there has been severe criticism of the policy from time to time over the years.

It is self evident that a policy such as this benefits the grain growers at the expense of the grain users and gives rise to questions of equity as between the two groups. The authors, however, were not concerned with equity considerations; rather were they concerned with the magnitudes of the relative "trade-offs", i.e., did the monetary advantage to the grain growers outweigh the disadvantages to the grain users, and what was the overall effect on the national income?

To resolve these questions an exercise was performed using the 1964 Agricultural Irish input–output table [1] and the results were published in the Economic and Social Review. [2] This chapter is a rearrangement of the latter paper in which an attempt is made to answer two questions:

(a) the effects on the economy of substituting home grown grains for some imported grains,

(b) the effects of the converse situation, where most of the required grains are imported.

The first part of this chapter gives in detail the process of investigating the effects denoted at (a) above, by using a miniature model of seven productive sectors. The second part of the chapter analyses the effects of (a) and (b) for the 1964 structure of Irish Agriculture and associated industries as published in [1]. Results for four sets of conditions, referred to as Exercises (1) to (4), are obtained.

PART 1

SUBSTITUTION OF HOME-GROWN FOR IMPORTED GRAIN, WITH VARYING EXPORTS OF BEEF, MILK PRODUCTS, MUTTON AND LAMB, FOR A 7-SECTOR MODEL

The Transactions Matrix

Suppose we have the simple economy illustrated by the 7-sector transactions Table 8.1 following. There are four Agricultural sectors, namely, Cattle, Dairying, Grain Growing and Pasture; and there are only three further sectors in the inter-industry complex, namely, Grain Milling/Animal Feed, Cattle Slaughter, Milk Processing. Final demand consists of Exports and Other Final.

The Primary Input rows are Imports of Grain, Other Imports, and Income Arising. (We ignore Rates, Subsidies and Depreciation). All transactions are shown in £000.

Following the Total Input row are two other rows; the first giving the Land Usage (arbitrary units) per £1 of output of the sectors directly using land, namely Grain Growing and Pasture; the second row gives the total land used by the system, the aggregate of the latter row (19 units) being the total available supply of land. Note that Cattle and Dairying, in the economy here considered, do not directly use land; their indirect use will appear below in the outcome of the matrix inversion, etc., see Table 8.1.

The direct input or technical coefficients to the seven productive sectors are given in Table 8.2.

These coefficients are obtained in the usual way by dividing the entries in each column of Table 8.1 by the column totals. Thus every entry in Column 1 is divided by 100, every entry in Column 4 is divided by 25 and so on for the other columns.

Adjustment of the Technical Coefficients

To replace imported by home-grown grain, the following assumptions are made:

(1) 11 tons of home-grown grain are equivalent to 10 tons of imports.

(2) In the initial situation, when there is both imported and home-grown grain on the market, the price of the home-grown produce is 20 per cent higher than that of the imports.

(3) Grain growers are given an extra £0·1 per £1 output at original prices to entice them to increase production of home-grown grains.

(4) Home Consumption of the products of all sectors, and exports of live cattle, are to remain unchanged, but exports of cattle slaughter and milk processing are allowed to vary. This implies that the columns of "Other Final" demands and the exports of sector 1 as given in Table 8.1 are left unchanged.

The first three of these assumptions involve adjusting the technical coefficients in Table 8.2. These adjustments are made as follows, see Table 8.3.

TABLE 8.1 7-Sector Transactions Table (£000)

SECTORS	Inter-Industry								Final Demand			
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	
	Cattle	Dairying	Grain Growing	Pasture	Grain Milling/ Animal Feed	Cattle Slaughter	Milk Processing	Total Inter-Industry	Exports	Other Final	Total Final	Total Output
1. Cattle	–	16	–	–	–	26	–	42	50	8	58	100
2. Dairying	35	–	–	–	–	–	40	75	–	25	25	100
3. Grain Growing	1	1	2	–	14	–	–	18	–	2	2	20
4. Pasture	15	10	–	–	–	–	–	25	–	–	–	25
5. Grain Milling/A.F.	2	2	–	–	–	–	–	4	–	36	36	40
6. Cattle Slaughter	–	–	–	–	–	5	–	5	20	15	35	40
7. Milk Processing	–	–	–	–	–	–	–	–	20	30	50	50
Total Inter-Industry	53	29	2	–	14	31	40	169	90	116	206	375
PRIMARY INPUTS												
Imports of Grain	–	–	–	–	10	–	–	10	–	–	–	10
Other Imports	1	2	3	8	5	7	2	28	–	–	–	28
Income Arising	46	69	15	17	11	2	8	168	–	–	–	168
Total Primary	47	71	18	25	26	9	10	206	–	–	–	206
Total Input	100	100	20	25	40	40	50	375	90	116	206	581
Land Usage per £1 of output	–	–	0·2	0·6	–	–	–	–	–	–	–	–
Total	–	–	4	15	–	–	–	19	–	–	–	19

TABLE 8.2 Technical coefficients based on Table 8.1

SECTORS	1	2	3	4	5	6	7
1. Cattle	–	0·16	–	–	–	0·650	–
2. Dairying	0·35	–	–	–	–	–	0·80
3. Grain Growing	0·01	0·01	0·10	–	0·350	–	–
4. Pasture	0·15	0·10	–	–	–	–	–
5. Grain Milling	0·02	0·02	–	–	–	–	–
6. Cattle Sl.	–	–	–	–	–	0·125	–
7. Milk Processing	–	–	–	–	–	–	–
PRIMARY INPUTS							
8. Imports of Grain	–	–	–	–	0·250	–	–
9. Other Imports	0·01	0·02	0·15	0·32	0·125	0·175	0·04
10. Income Arising	0·46	0·69	0·75	0·68	0·275	0·050	0·16
Total Input	1·00	1·00	1·00	1·00	1·000	1·000	1·00
Land Usage	–	–	0·2	0·6	–	–	–

The import coefficient of 0·250 in Row 8 of Column 5 is first increased by the ratio 11/10 (assumption (1) above) and then by 6/5 (assumption (2) above) so that it becomes 0·330 (i.e. 0·250 x 11/10 x 6/5 = 0·330). This adjusted coefficient is then removed from Row 8 and added on to 0·350 in Row 3 of column 5 to give a coefficient of 0·680 for native grain sold to milling. The latter coefficient is greater by 0·080 than the sum of the two former coefficients which it replaces, i.e. 0·680 – (0·250 + 0·350) = 0·080, hence an adjustment of –0·080 is necessary in Column 5 to keep the column sum unity.

Similarly for Column 3. From assumption (3) above grain growers are given an extra £0·1 per £1 output at original prices, therefore it is necessary to increase the coefficient of 0·75 for income arising in this column by 0·1 to give a coefficient of 0·85 for this item. Following from this, an adjustment of –0·10 is necessary in Column 3 to keep the column sum unity. This is the c_j adjustment mentioned in Part 2 of this chapter. The adjusted matrix of technical coefficients is given in Table 8.3.

The $(I - A)$ matrix and its inverse $(I - A)^{-1}$ matrix

The $(I - A)$ matrix derived from the first seven rows and columns of Table 8.3 is given in Table 8.4. The entries in this table are derived in the usual way by subtracting each diagonal entry in Table 8.3 from 1·0 and applying minus signs to all the other entries.

The inverse of the $(I - A)$ matrix in Table 8.4 is given in the first seven rows and columns of Table 8.5.

The column of outputs in Table 8.5 is obtained by multiplying in turn the

TABLE 8.3 Adjusted Technical Coefficients

SECTORS	(1)	(2)	(3)	(4)	(5)	(6)	(7)
1. Cattle	–	0·16	–	–	–	0·650	–
2. Dairying	0·35	–	–	–	–	–	0·80
3. Grain Growing	0·01	0·01	0·10	–	0·680	–	–
4. Pasture	0·15	0·10	–	–	–	–	–
5. Grain Milling	0·02	0·02	–	–	–	–	–
6. Cattle Slaughter	–	–	–	–	–	0·125	–
7. Milk Processing	–	–	–	–	–	–	–
PRIMARY INPUTS							
8. Other Imports	0·01	0·02	0·15	0·32	0·125	0·175	0·04
9. Income Arising	0·46	0·69	0·85	0·68	0·275	0·050	0·16
10. Total Primary	0·47	0·71	1·00	1·00	0·400	0·225	0·20
11. Adjustment			–0·10		–0·080		
Total Input	1·00	1·00	1·00	1·00	1·000	1·000	1·00
Land Usage	–	–	0·2	0·6	–	–	–

TABLE 8.4 $(I-A)$ Matrix Derived from Table 8.3 with a further column for known Final Demands

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	Known Final Demand*
								£000
1.	1·00	–0·16	–	–	–	–0·650	–	58
2.	–0·35	1·00	–	–	–	–	–0·80	25
3.	–0·01	–0·01	0·90	–	–0·68	–	–	2
4.	–0·15	–0·10	–	1·00	–	–	–	–
5.	–0·02	–0·02	–	–	1·00	–	–	36
6.	–	–	–	–	–	0·875	–	15
7.	–	–	–	–	–	–	1·00	30

* For explanation of Known Final Demand entries see assumption (4) given previously.

entries in each row of the inverse matrix by the entries in the column of known final demands in Table 8.4 and summing the results. The output of 81·5 given in the first row of Table 8.5 is obtained as follows:

TABLE 8.5 Inverse of $(I-A)$ Matrix in Table 8.4 together with Sector Outputs required directly and indirectly by Known Final Demands

	Inverse Matrix $(I-A)^{-1}$							Derived Sector Outputs
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	
1.	1·059321	0·169491	0·0	0·0	0·0	0·786925	0·135593	81·5
2.	0·370763	0·059321	0·0	0·0	0·0	0·275424	0·847457	77·5
3.	0·037500	0·032222	1·111111	0·0	0·755556	0·027857	0·025778	33·6
4.	0·195974	0·131356	0·0	1·0	0·0	0·145581	0·105085	20·0
5.	0·028602	0·024576	0·0	0·0	1·0	0·021247	0·019661	39·2
6.	0·0	0·0	0·0	0·0	0·0	1·142857	0·0	17·1
7.	0·0	0·0	0·0	0·0	0·0	0·0	1·0	30·0

(58 x 1·059321) + (25 x 0·169491) + (2 x 0·0) + (0 x 0·0) + (36 x 0·0) + (15 x 0·786925) + (30 x 0·135593) = 81·5 (correct to one decimal place). Similarly the output of 77·5 in the second row of the table is obtained by multiplying the second row of the inverse matrix by the same column of final demands, and so on for the other output entries.

By applying the technical coefficients from Table 8.3 to the column of derived outputs in Table 8.5 we can obtain the transactions table relevant to the adjusted system. The primary inputs from this transactions table are given in Table 8.6 with Land requirements derived by applying the Land Usage coefficients from either Table 8.2 or 8.3 to the derived outputs.

TABLE 8.6 Primary Inputs (£1000) and Land Requirements for Derived Outputs

Type of Primary	Derived Outputs							Total Primary Inputs
	81·5	77·5	33·6	20·0	39·2	17·1	30·0	
	Primary Inputs							
Other Imports	0·8	1·6	5·0	6·4	4·9	3·0	1·2	22·9
Income Arising	37·5	53·5	28·6	13·6	10·8	0·9	4·8	149·6
Adjustment	0·0	0·0	−3·4	0·0	−3·1	0·0	0·0	−6·5
Land Usage	0·0	0·0	6·7	12·0	0·0	0·0	0·0	18·7

The information given in Table 8.6 is not sufficient for the solution of our problem. We also need to know the primary input requirements per unit of the various kinds of final demands. These requirements are given in Table 8.7. They are obtained by multiplying in turn each column of the inverse matrix by a row of the primary input technical coefficients in Table 8.3 and summing the results. Thus the "other import" requirement per unit of final demand for the output of sector 1 is:
(0·01 x 1·059321) + (0·02 x 0·370763) + (0·15 x 0·037500) + (0·32 x 0·195974) + (0·125 x 0·028602) + (0·175 x 0·0) + (0·04 x 0·0) = 0·089920, see Table 8.7. Similarly the "other import" requirement of one unit of final demand for the output of Sector 2 is obtained by multiplying the row of "other import" technical coefficients by the second column of the inverse matrix: i.e. (0·01 x 0·169491) + (0·02 x 1·059321) . . . . + (0·04 x 0·0) = 0·072821 and so on for the "other import" requirements of the other sectors. The "income arising" for a unit of final demand for the output of sector 1 is:
(0·46 x 1·059321) + (0·69 x 0·370763) . . . . + (0·16 x 0·0) = 0·916117. The "adjustment" requirement for a unit of final demand for the output of sector 3 is: (0·0 x 0·0) + (0·0 x 0·0) − (0·10 x 1·111111) + (0·0 x 0·0) − (0·080 x 0·0) + (0·0 x 0·0) + (0·0 x 0·0) = −0·111111.

TABLE 8.7 Primary Inputs and Land requirements per unit of the various kinds of Final Demand

Type of Primary	(1)	(2)	(3)	(4)	(5)	(6)	(7)	Aggregate £000
Other Imports	0·089920	0·072821	0·166667	0·320000	0·238333	0·266798	0·098256	22·9
Income Arising	0·916117	0·932367	0·944444	0·680000	0·917222	0·737687	0·905894	149·6
Adjustment	−0·006038	−0·005188	−0·111111	0·0	−0·155555	−0·004485	−0·004151	−6·5
Land Usage	0·125085	0·085258	0·222222	0·600000	0·151111	0·092920	0·068206	18·7 (units)

The aggregate figures in this table (which are exactly the same as the total primary inputs in Table 8.6) are obtained by multiplying each row entry in Table 8.7 by the known final demand in Table 8.4 corresponding to that row. Thus the figure of 22·9 in the first row is obtained as follows:
(58 x 0·089920) + (25 x 0·072821) + (2 x 0·166667) + (0 x 0·320000) + (36 x 0·238333) + (15 x 0·266798) + (30 x 0·098256) = 22·9. Similarly for the other aggregate figures in Table 8.7.

The advantage of Table 8.7 for the purpose under review is that it shows indirect as well as direct requirements for the various primary inputs, whereas the technical coefficients in Table 8.3, which are based on sector outputs, show only the direct input requirements. It should be noted that, because of indirect effects, the land usage and the adjustments are spread over practically all of the sectors in Table 8.7, compared with Table 8.6 where these inputs occur only in the sectors where they are directly required.

Solution of Problem

The problem to be solved is that of finding the levels of sector outputs, income arising, exports etc. under conditions where native grain replaces imported grain. Grain growers get an extra £0·1 per £1·0 of output at original prices as an incentive with the following constraints:
(1) Total land area to remain at 19 units as originally,
(2) Cattle and Dairying to have their outputs in the same proportions, i.e. 1:1 as before.

These two contraints imply two unknowns which must be determined. Let these be B and M, the exports of beef and milk products respectively. These unknowns are the additional final demands for the outputs of Sectors 6 and 7 respectively, so that the new final demands are:

Final Demands
(£000)

Sector	Known	Additional	Total (new)
1	58	–	58
2	25	–	25
3	2	–	2
4	–	–	–
5	36	–	36
6	15	B	$15+B$
7	30	M	$30+M$

By multiplying the new columns of final demands by the inverse matrix we can obtain the outputs relevant to these final demands. Thus the total output

required from Sector 1 (cattle) is that required by the known final demands given in Table 8.5 (81·5) plus B times element 1 of Column 6 of inverse matrix plus M times element 1 of Column 7 of inverse i.e.

(a) $$81{\cdot}5496 + 0{\cdot}786925B + 0{\cdot}135593M \qquad (8.1)$$

Similarly the output of Sector 2 (dairying) is:

(b) $$77{\cdot}5423 + 0{\cdot}275424B + 0{\cdot}847457M \qquad (8.2)$$

Since we have specified that cattle and dairying must be in proportion of 1:1 these two outputs must be equal, and so our first equation is

$$81{\cdot}5496 + 0{\cdot}786925B + 0{\cdot}135593M = 77{\cdot}5423 + 0{\cdot}275424B + 0{\cdot}847457M \qquad (8.3)$$

In order to find unique values for B and M a second equation is needed. This is derived from the land use specification which states that total land is to remain at 19 units as originally. As for equation (8.3), total land usage is made up of usage based on known final demands as given in Table 8.7 (18·7 units) plus B and M multiplied by their respective total land requirements per unit from Columns 6 and 7 of Table 8.7 also, i.e.

$$19{\cdot}0 = 18{\cdot}7108 + 0{\cdot}092920B + 0{\cdot}068206M \qquad (8.4)$$

Solving equations (8.3) and (8.4) we obtain $B = -0{\cdot}6676$
$M = 5{\cdot}1496$

As B and M are total exports of Beef and Milk respectively, which were 20·0 and 20·0 in Table 8.1, the negative value of B means that a small amount of beef has to be imported. The original and new final demands are shown in Table 8.8.

New Values for Sector Outputs and Primary Inputs

Once B and M have been determined, the outputs and primary inputs for the new situation are obtained by applying B and M to Columns 6 and 7 of the inverse matrix (Table 8.5), and to the corresponding primary input requirement per unit of final demand as given in Table 8.7. The results of this exercise are then added to the outputs and primary inputs generated by the known final demands, which are given in Tables 8.5 and 8.6 respectively, to obtain the overall results as shown in Table 8.9.

It should be noted from Table 8.9 that outputs of cattle and dairying are equal, as required by equation (8.3), and that the total area of land used is unchanged at 19 units. The land used could also be obtained by applying the technical coefficients for land use from either Tables 8.2 or 8.3 to the new total outputs of grain growing and pasture in Table 8.9.

(i.e. $0{\cdot}2 \times 33{\cdot}71 + 0{\cdot}6 \times 20{\cdot}44 = 19{\cdot}0$)

TABLE 8.8 Original and new Final Demands (£000)

Sectors	Original			New			Difference		
	Exports (1)	Other (2)	Total (3)	Exports (4)	Other (5)	Total (6)	Exports (7)	Other (8)	Total (9)
1. Cattle	50	8	58	50	8	58	–	–	–
2. Dairying	–	25	25	–	25	25	–	–	–
3. Grain Growing	–	2	2	–	2	2	–	–	–
4. Pasture	–	–	–	–	–	–	–	–	–
5. Grain Milling	–	36	36	–	36	36	–	–	–
6. Cattle Slaughter	20	15	35	−0·67(*B*)	15	14·33	−20·67	–	−20·67
7. Milk Processing	20	30	50	5·15(*M*)	30	35·15	−14·85	–	−14·85
Total	90	116	206	54·48	116	170·48	−35·52	–	−35·52

TABLE 8.9 New Values for Sector Outputs and Primary Inputs.

Sectors	Col. 6 of Inverse	Col. 7 of Inverse	$B(-0{\cdot}6676)$ multiplied by (1)	$M(5{\cdot}1496)$ multiplied by (2)	Output required for known Final Demands	New Outputs (3) + (4) + (5)	Original Outputs	Difference: (6) less (7)
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
					£000	£000	£000	£000
1. Cattle	0·786925	0·135593	−0·53	0·70	81·55	81·72	100·00	−18·3
2. Dairying	0·275424	0·847457	−0·18	4·36	77·54	81·72	100·00	−18·3
3. Grain Growing	0·027857	0·025778	−0·02	0·13	33·60	33·71	20	13·7
4. Pasture	0·145581	0·105085	−0·10	0·54	20·00	20·44	25	−4·6
5. Grain Milling	0·021247	0·019661	−0·01	0·10	39·20	39·29	40	−0·7
6. Cattle Slaughter	1·142857	0·0	−0·76	0·00	17·10	16·34	40	−23·7
7. Milk Processing	0·0	1·000000	−0·00	5·15	30·00	35·15	50	−14·8
Primary Inputs	Input Multipliers Table 8.7 (Col. 6)	(Col. 7)			Inputs for Known Finals	New Inputs	Original Inputs	
Imports of Grain	–	–	–	–	–	–	10	−10·0
Other Imports	0·266798	0·098256	−0·18	0·51	22·90	23·23*	28	−4·8
Income Arising	0·737687	0·905894	−0·49	4·66	149·60	153·77	168	−14·2
Adjustment	−0·004485	−0·004151	+0·00	−0·02	−6·50	−6·52	–	−6·5
Land Usage	0·092920	0·068206	−0·06	0·35	18·71	19·00	19	Nil

* Excludes negative export of beef which is netted out of total final demand for beef.

As expected the output of grain growing is up from 20 in the original table to 33·7 here. Total imports are down by 14·8 (i.e. 10 + 28 − 0 − 23·2 = 14·8), but if we look at Table 8.8 we see that total exports are down by 35·52, hence there is a decrease in exports less imports. As there is also a decrease in income arising, the economy would lose by changing over completely to home grain growing.

The adjustment can be regarded as a necessary subsidy paid by the government to keep prices stable. It is paid to producers as a negative production cost, and if there were no adjustment prices would have to rise in order to cover the increased income postulated for grain growers. These price effects are discussed in the following section, but in order to understand how they are calculated it is convenient to have the complete transactions table for the new situation. This table is prepared by applying the technical coefficients (including the adjustment and land usage coefficients) from Table 8.3 to the new total outputs (i.e. the new outputs in Column 6 of Table 8.9). Table 8.10 is the new transactions table at original prices.

Price Changes in the Absence of Adjustment

The vector of Total Primary Inputs (Row 10 of Table 8.3), when post-multiplied by the inverse matrix, gives the changes in price levels arising from the price incentive to grain growing, and substitution of the more expensive native grain for imported grain. When this operation is carried out the resulting price changes are:

Row	(1)	(2)	(3)	(4)	(5)	(6)	(7)
Price Change	1·006038	1·005188	1·111111	1·0	1·155555	1·004485	1·004151

It is of interest to note that these price ratios exceed unity by amounts exactly equal to, and of opposite sign to, the adjustment coefficients per unit of final demand given in Table 8.7.

As can be seen, the actual numerical price ratios for the seven sectors are found to be hardly different from unity except for Grain Growing (3) and Grain Milling (5) where the price changes are 11 and 16 per cent respectively. Of the 11 per cent, 10 per cent is due to the grain growing incentive; while of the 16 per cent, 8 per cent is due to the substitution of native for imported grain at original prices, and a further 7 per cent is due to the increased prices which come about as a result of the grain-growing incentive. Thus the direct observable first-order causes explain most of the computed price change, while the residue of each change (positive or negative) is due to second, third and higher order effects.

TABLE 8.10 Transactions Table for New Situation at Original Prices (£000)

SECTORS	INTER-INDUSTRY								FINAL DEMAND			
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	
	Cattle	Dairying	Grain Growing	Pasture	Grain Milling	Cattle Slaughter	Milk Processing	Total Inter.	Exports	Other Final	Total Final	Total Output
1. Cattle	–	13·09	–	–	–	10·63	–	23·72	50	8	58	81·72
2. Dairying	28·60	–	–	–	–	–	28·12	56·72	–	25	25	81·72
3. Grain Growing	0·82	0·82	3·37	–	26·70	–	–	31·71	–	2	2	33·71
4. Pasture	12·26	8·17	–	–	–	–	–	20·43	–	–	–	20·44
5. Grain Milling/A.F.	1·64	1·64	–	–	–	–	–	3·28	–	36	36	39·29
6. Cattle Slaughter	–	–	–	–	–	2·01	–	2·01	−0·67	15	14·33	16·34
7. Milk Processing	–	–	–	–	–	–	–	–	5·15	30	35·15	35·15
Total Inter-Industry	43·32	23·72	3·37	–	26·70	12·64	28·12	137·87	54·48	116	170·48	308·37
8. Other Imports	0·81	1·63	5·06	6·54	4·92	2·87	1·40	23·23	–	–	–	23·23
9. Income Arising	37·59	56·37	28·65	13·90	10·81	0·83	5·62	153·77	–	–	–	153·77
10. Adjustment	–	–	−3·37	–	−3·14	–	–	−6·51	–	–	–	−6·51
11. Total Inputs	81·72	81·72	33·71	20·44	39·29	16·34	35·15	308·37	54·48	116	170·48	478·86
12. Land Usage			6·74	12·26				19·00				19

Note – Due to rounding errors the additions of the various entries do not always give the correct total outputs or inputs.

Transactions Table at New Prices

If the adjustment row of the new transactions table (Table 8.10) is deleted, and the Primary Input rows including "income arising" are left unchanged, we can obtain a transactions table at new prices by multiplying the entries in each production row of Table 8.10 by the corresponding price change shown above. Thus the entries in the first row of Table 8.10 are multiplied by 1·006038, those in the second by 1·005188, and so on for the others. Table 8.11, which has been derived in this way from Table 8.10, shows that though aggregate income arising remains the same in the two tables both internal and export values are higher in the former than in the latter as a result of the price increases. The same physical flows of products as in Table 8.10 still apply, but now there is no Primary Input row of Adjustment amounts to absorb the grain growers' incentive and the extra expense of native grain. Consequently the extra cost per physical unit must be absorbed by price increases. In a situation such as this, increased export prices could have serious repercussions on demand, and so it may be deemed necessary to subsidize certain exports so as to keep their prices constant, while at the same time allowing the higher prices on the home market. The amount of subsidy required for this purpose is obtained simply by deducting the export value of each sector at original prices from the corresponding value at the new prices and summing the result. For our present example the sectoral subsidies required to keep export prices constant are as follows:

£000

Sector	1	2 3 4 5 6	7	Total
Subsidy	0·30		0·02	0·32

These figures are obtained by deducting the entries in the Export column of Table 8.10 from the corresponding entries in the Export column of Table 8.11.

It is worth observing that the 6·51 increase in Income Arising, absorbed by the Adjustment row of Table 8.10, is exactly passed on to the Final Demand in Table 8.11. Final Demands in the latter table pay 176·99 for the same physical flows of goods as cost them 170·8 in Table 8.10, the increase in cost in aggregate being exactly 6·51. This property of Input–Output Tables in general can be readily explained by reference to Table 8.11. The aggregate Total Input of productive Sectors 1 to 7 (319·35) is by definition equal to the Total Output of these same 7 productive Sectors (319·35). If we net out of both schemes the aggregate total inter-industry (or intermediate) absorption of the flows (142·35), this leaves aggregate flow to Final Demand equal to aggregate Primary Input, both amounts being 177·00. Thus, in general, any change in Primary Inputs to productive sectors will produce an equal change in Final Demands of these sectors.

TABLE 8.11 Transactions Table for New Situation at New Prices (£000)

Inputs ↓	INTER-INDUSTRY								FINAL DEMAND			
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
	Cattle	Dairying	Grain Growing	Pasture	Grain Milling	Cattle Slaughter	Milk Processing	Total Inter.	Exports	Other Final	Total Final	Total Outputs
1. Cattle	–	13·17	–	–	–	10·69	–	23·86	50·30	8·05	58·35	82·21
2. Dairying	28·75	–	–	–	–	–	28·27	57·02	–	25·13	25·13	82·15
3. Grain Growing	0·91	0·91	3·74	–	29·66	–	–	35·22	–	2·22	2·22	37·44
4. Pasture	12·26	8·17	–	–	–	–	–	20·43	–	–	–	20·44
5. Grain milling/A.F.	1·90	1·90	–	–	–	–	–	3·80	–	41·60	41·60	45·40
6. Cattle Slaughter	–	–	–	–	–	2·02	–	2·02	–0·67	15·07*	14·40	16·42
7. Milk Processing	–	–	–	–	–	–	–	–	5·17	30·12	35·29	35·29
Total Inter-Industry	43·82	24·15	3·74	–	29·66	12·71	28·27	142·35	54·80	122·19	176·99	319·35
Other Imports	0·81	1·63	5·06	6·54	4·92	2·87	1·40	23·23	–	–	–	23·23
Income Arising	37·59	56·37	28·65	13·90	10·81	0·83	5·62	153·77	–	–	–	153·77
Total Inputs	82·21	82·15	37·44	20·44	45·40	16·42	35·29	319·35	54·80	122·19	176·99	496·35

* Note – Due to rounding errors the additions of the various entries do not always give the correct total outputs or inputs.

PART 2

SUBSTITUTION BETWEEN HOME-GROWN AND IMPORTED GRAIN, FOR 1964 IRISH AGRICULTURE AND ASSOCIATED INDUSTRIES

More Home-Grown Grain, With Less Imports

The native grains in question are wheat, oats, and feeding barley. Exercises (1) and (2), which investigate this situation, assumes that a minimum amount of hard wheat, namely one-quarter of the flour-milling requirement (amounting to 91 150 tons for 1964) would have to be imported. We made the further assumption that if the extra growing of wheat, oats, and barley was to take place, the income arising in these three sectors would have to be increased substantially per unit of output, in order to entice farmers to substitute grain-growing for other farming enterprises. An extra £0·2 per £1 of output at original 1964 prices was considered an adequate incentive. Although we think this extra 0·2 per unit to be realistic, others might choose a different level of incentive.

The area of land used for all farming activity was held constant, by means of total Land Annuities* and annuity coefficients for the various crops; thus if grain growing increased, some other sectors would necessarily be reduced.

We realise that for Exercises (1) and (2) there will be price increases, due to native grains having a higher unit cost than imported ones and also due to the extra profit margins on the home-grown cereals. At these prices we assume that products produced from native grain could not be exported unless a subsidy were applied to keep export prices at their original 1964 levels; the subsidy possibly being an acreage payment.

Substitution Rates

For imported wheat replaced by native wheat we used a 1:1 tonnage substitution rate. For feed grain, however, we used the standard substitution rates as published in the Department of Agriculture Leaflet Number 2 "Food for Livestock"

i.e. when Barley = 10,	Milo = 9.5	
Wheat = 10,	Oats = 12.0	
Maize = 9.5,	Wheat Offals = 13.0	(8.5)

Less Home-Grown Grains, With More Imports

Exercises (3) and (4), which investigate this situation, have all grains imported, except (i) oats grown and consumed on farms without process of sale, and (ii) malting barley. If this situation were to come about there would be

* Land annuities are a form of rent paid by farmers to the Irish Land Commission. By keeping the sum of all the annuities constant the total land area of the State is forced into use.

more land available for other farming activities. Also the extra imported grains might bring about some reductions in product price levels.

A summary of quantities and values for production, imports, and disposal of grains in 1964 is given in Table 8.12 following.

Methodology

In the 1964 Acricultural Input–Output Transactions Table, as published, the flows along some of the rows contained both domestic products and imports of similar commodities. It was first of all necessary, before attempting any cereal substitutions, to separate out the domestic and import content of each non-zero entry in all such rows. Eight cereal items were considered wholly or partly replaceable by domestic items, namely, wheat, oats, barley, maize, milo, wheaten flour, bran/pollard, maize meal. All the imports included in any of the rows of the original 1964 table were brought down to the Primary Input section, leaving purely domestic flows along the Inter-Industry rows.

The 8 cereal items were preserved in 8 separate rows of the Primary Inputs while the other competitive imports (not under review here) were aggregated into a single row. There were two further rows for non-competitive imports, i.e. one for the estimated import content of fertilizers and one for the estimated import content of other expenses.

After arranging the transactions table as indicated, the direct input coefficients (also called technical coefficients) were calculated, by dividing the Total Input cost of each column into each entry in that column, these technical coefficients adding to unity in each column.

EXERCISE (1)

SUBSTITUTION OF HOME-GROWN FOR IMPORTED GRAIN, WITH VARYING EXPORTS OF BEEF, MILK PRODUCTS, MUTTON, AND LAMB

General Assumptions

Replace wholly or partially, as indicated above, the 8 cereal import items by domestic equivalents; increase the income-arising coefficient in wheat, barley,

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REFERENCES – CHAPTER 8

[1] O'Connor, R., with Breslin, M., "An Input–Output Analysis of the Agricultural Sector of the Irish Economy in 1964", *Paper No. 44 of the Economic and Social Research Institute,* Dublin (1968)

[2] Henry, E.W., and O'Connor, R., 'Irish Agriculture and associated industries 1964: 'Effects of various cereal policies', *Economic and Social Review,* Vol. 1, No. 3, Dublin (April 1970).

TABLE 8.12 Production, Imports, and Disposal of Certain Cereals in 1964 on a Dried Grain Basis

Cereal	Home Production	Imports	Total Supply	Disposal					
				Human Consumption	Animal Feed			Seed	Changes in Stocks and Exports
					Unsold	Sold (as such)	Compound Feed		
	Quantity (000 tons)								
Wheat	240·6	168·4	409·0	358·2	21·4	nil	55·6	17·2	−43·4
Oats	292·7	13·6	306·3	18·9	216·9	26·9	19·5	23·2	0·9
Barley (Feeding and Malting)	487·9	0·4	488·3	111·2	122·0	47·0	223·6	27·2	−42·7
Maize & Milo	nil	193·0	193·0	3·2	nil	43·7	125·7	nil	20·4
Wheat Offals	106·3	106·6	212·9	nil	nil	62·0	140·9	nil	10·0
TOTAL	1 127·5	482·0	1 609·5	491·5	360·3	179·6	565·3	67·6	−54·8
	Value (£000)								
Wheat	7 280	4 450	11 730	10 836	258	nil	1 463	562	−1 389
Oats	2 901	381	3 282	254	1 460	608	520	421	19
Barley	10 181	13	10 194	2 919	1 272	1 295	5 031	639	−962
Maize & Milo	nil	4 320	4 320	72	nil	1 006	2 799	nil	443
Wheat Offals	2 237	2 099	4 336	nil	nil	1 354	2 778	nil	204
TOTAL	22 599	11 263	33 862	14 081	2 990	4 263	12 591	1 622	−1 685

and oats columns by 0·2. In the replacement, soft wheat imports were replaced by native wheat, imported oats by native oats, and imported barley by native barley. Also imported flour was replaced by native flour; maize, milo, bran/pollard, and maize meal were all replaced by native barley, the latter first being milled or compounded, except for a few smallish bran/pollard import amounts going directly to animals. The latter were replaced by unmilled native barley. Part 1 illustrated numerically, via a 7-sector model, the different stages of reaching the final solution to a problem of this type. The general method of reaching such a solution is shown below.

Adjustment of Technical Coefficients

The Department of Agriculture substitution rates referred to above, together with prices per ton for domestic and import items (in original 1964 Table) were used to adjust the technical coefficients of the domestic substitutes. The following scheme illustrates the transformations required, with row subscripts 9, 10, and 11 relating to the row numbers in the table and j indicating any relevant column.

Technical coefficients before substitution of domestic for imports (Column j)		Technical coefficients after substitution of domestic for imports (Column j)	
Inter-Industry		*Inter-Industry*	
Native wheat	a_{9j}	Native wheat	$a_{9j} + \lambda_w w_j$
Native oats	a_{10j}	Native oats	a_{10j}
Native barley	a_{11j}	Native barley	$a_{11j} + \lambda_b b_j + \lambda_m m_j$
*Primary**		*Primary**	
Imported wheat	w_j	Imported wheat	nil
Imported barley	b_j	Imported barley	nil (8.6)
Imported milo	m_j	Imported milo	nil
		Adjustment coefficient	c_j
Total input	1·0	Total	1·0

The parameters λ_w, λ_b and λ_m combine the feed equivalent tonnage substitution rates with the original price-per-ton ratios, for wheat, barley, and milo respectively, e.g. for barley replacing milo:

* For convenience of exposition the five other grain import items have been omitted.

$$\lambda_m = \frac{\text{substitution rate for barley}}{\text{substitution rate for milo}} \times \frac{\text{price per ton home grown barley}}{\text{price per ton of milo}} \tag{8.7}$$

and similarly for the other λ's.

The "Adjustment coefficient" c_j is put in to make the column sum still unity. With c_j included, all prices are still at their original 1964 levels and we can work within the old price system and find the total adjustment necessary to keep prices stable. If, however, the adjustment coefficient be omitted, the quantity results will still be the same, but now price levels will change since the column sums are no longer unity; these changes in price levels can be calculated, and are given by the row vector:

$$[p_1, p_2 \ldots . p_n] = [\pi_1, \pi_2 \ldots . \pi_n] [I - A]^{-1} \tag{8.8}$$

where $[\pi_1, \pi_2 \ldots . \pi_n]$ are the sums of the Primary Inputs for each column excluding Adjustment Coefficient row but including any extra profit margin (such as 0·2 on grains); and A is the matrix of technical coefficients, after the substitution at original prices of native for imported grains.

In the Input–Output scheme, any vector of domestic outputs X required for a specified Final Demand Y is given by

$$X = (I - A)^{-1} Y \tag{8.9}$$

where A is the inter-industry matrix of technical coefficients (here after domestic substitution for imports).

I is the Unit Matrix, and

$(I - A)^{-1}$ is the inverse of $(I - A)$. Each column j of the inverse gives the total amounts of all sectors' outputs required for one unit of Final Demand for domestic commodity j.

Assumptions concerning Land Use and Final Demands

The extra use of land for grain growing requires necessary reductions in land available for livestock. The Land Annuity coefficients were used to measure the amount of land per unit of production of relevant items. It was decided also to keep Other Industries, Personal Consumption, and Stock Changes identical with original values (at original prices) apart from minor adjustments due to replacement of imported cereals by domestic equivalents. It was furthermore decided to keep at original levels and prices the exports of live cattle, live sheep, live horses, and the small amount of exports of whole and skim milk.

Thus the three exports, being the only parts of Final Demand to be changed, were

(1) Mutton/Lamb and a small amount of Horse Meat (L)

(2) Beef (B)
(3) Milk Processing (M)
These were denoted L, B and M respectively.

The total final demand vector for Exercise (1) was therefore subdivided into

(*a*) a vector of *known values* which included Domestic Consumption and Stock Changes for all sectors; and exports for all sectors except Cattle Slaughter, Sheep/Horse Slaughter, and Milk Processing, and

(*b*) a second vector having zero elements in all rows except in those for Cattle Slaughter, Sheep/Horse Slaughter, and Milk Processing, where the three letters L, B and M were entered respectively. These letters represented the exports of the latter three sectors which are unknown.

If these were known the resultant total domestic output X would be given by:

$$X = [I - A]^{-1} \begin{bmatrix} \text{Column} \\ \text{Vector} \\ \text{of known} \\ \text{Values} \end{bmatrix} + B \begin{bmatrix} \text{Cattle} \\ \text{Slaughter} \\ \text{Column} \\ \text{of Inverse} \end{bmatrix} + L \begin{bmatrix} \text{Sheep,} \\ \text{Horse} \\ \text{Slaughter} \\ \text{Column} \\ \text{of Inverse} \end{bmatrix} + M \begin{bmatrix} \text{Milk} \\ \text{Process-} \\ \text{ing} \\ \text{Column} \\ \text{of Inverse} \end{bmatrix}$$

In fact:

X_1 (cattle raising) new output is given by cattle output required by known final demands

$+ B$ (element 1 of Cattle Slaughter column of Inverse)
$+ L$ (element 1 of Sheep, Horse Slaughter column of Inverse) (8.10)
$+ M$ (element 1 of Milk Processing column of Inverse)

Similarly for X_2 (dairying) and so on.

For each column j of the $(I - A)$ Inverse we can also find a "Total" requirement coefficient for any Primary Input such as Land Annuities. This coefficient gives the complete direct and indirect amounts of the Primary Inputs required by one unit of Final Demand for domestic output j; it is given by the scalar product of column j of the Inverse and the row of Direct Input coefficients for the Primary in question (see text preceding Table 8.7 of Part 1). Such total requirement coefficients were calculated for Land Annuities and certain other primary inputs.

Determination of Unknowns, B, L and M

To determine values for the three unknowns, B, L and M, three equations are required. For Exercise (1) the three equations were based on the following assumptions:

(1) Cattle (Sector 1) and Dairying (Sector 2) outputs to be in the same proportions as for 1964 originals, i.e. 88 109 / 88 754.

(2) Dairying (Sector 2) and Milk Processing (Sector 24) to be in proportion of

1964 original outputs, i.e. 88 754 / 43 451.

(3) Land Annuity: total specified to be unchanged.

This implies constant area of land in use.

On the basis of these assumptions the three equations were derived as follows. From (8.10) above:

$$(a) \qquad 72\,615{\cdot}2 + 0{\cdot}704\,542B + 0{\cdot}000\,067L + 0{\cdot}189\,982M = X_1 \qquad (8.11)$$

where:

72 615·2 is the output of cattle required by the known final demands obtained by pre-multiplying the vector of known final demands by the inverse matrix,

0·704 542 is element 1 of Cattle Slaughter column of inverse,

0·000 067 is element 1 of Sheep/Horse Slaughter column of inverse,

0·189 882 is element 1 of Milk Processing column of inverse, and

X_1 is the new derived output of the Cattle sector.

similarly:

$$(b) \qquad 63\,570{\cdot}6 + 0{\cdot}000\,901B + 0{\cdot}000\,219L + 1{\cdot}208\,733M = X_2 \qquad (8.12)$$

where:

63 570·6 is the output of Dairying required by the known final demands obtained in the same way as output of Cattle at (*a*) above,

0·000 901 is element 2 of Cattle Slaughter column of inverse,

0·000 219 is element 2 of Sheep/Horse Slaughter column of inverse,

1·208 733 is element 2 of Milk Processing column of inverse, and

X_2 is the new derived output of the Dairying sector.

Now from assumption (1) above

$$\frac{X_1}{X_2} = \frac{88\,109}{88\,754} \qquad (8.13)$$

hence our first equation is, via (*a*) and (*b*) substituted in (8.13)

$$88\,754 \times [72\,615{\cdot}2 + 0{\cdot}704\,542B + 0{\cdot}000\,067L + 0{\cdot}189\,982M] =$$

$$88\,109 \times [63\,570{\cdot}6 + 0{\cdot}000\,901B + 0{\cdot}000\,219L + 1{\cdot}208\,733M]$$

which is

$$0{\cdot}708\,799B - 0{\cdot}000\,152L - 1{\cdot}017\,360M = -9\,576{\cdot}2 \qquad (8.14)$$

The second equation was derived using equation (*b*) above and the following equation (*c*) for milk processing

$$(c) \qquad 22\,581{\cdot}0 + 0{\cdot}000\,746B + 0{\cdot}000\,181L + 1{\cdot}001\,704M = X_{24} \qquad (8.15)$$

where

22 581·0 is the output of Milk Processing required by the known final demand, obtained in the same way as the output of Cattle at (*a*) above,
0·000 746 is element 24 of Cattle Slaughter column of inverse,
0·000 181 is element 24 of Sheep/Horse Slaughter column of inverse,
1·001 704 is element 24 of Milk Processing column of inverse,
and X_{24} is the new derived output of the Milk Processing sector.

Now from assumption (2) above

$$\frac{X_2}{X_{24}} = \frac{88\,754}{43\,451} \tag{8.16}$$

The second equation is

$$88\,754\ [22\,581{\cdot}0 + 0{\cdot}000\,746B + 0{\cdot}000\,181L + 1{\cdot}001\,704M] =$$
$$43\,451\ [63\,570{\cdot}6 + 0{\cdot}000\,901B + 0{\cdot}000\,219L + 1{\cdot}208\,733M]$$

which is

$$0{\cdot}000\,063B + 0{\cdot}000\,151L + 0{\cdot}837\,370M = 17\,446{\cdot}1 \tag{8.17}$$

The third equation, which is based on the land annuity restriction, is as follows:

$$2547{\cdot}1 + 0{\cdot}011\,993B + 0{\cdot}028\,146L + 0{\cdot}014\,589M = 3045{\cdot}1 \tag{8.18}$$

where:

2547·1 = the value of annuities associated with all the unchanged sectors of the model, while
(0·011 993), (0·028 146) and (0·014 589) are the annuity total requirement coefficients of cattle slaughter, sheep slaughter and milk processing respectively, derived as explained above.
3045·1 = the land annuity total for the State in the original table which is left unchanged.

By solving these equations, the value of B, L and M were found to be (in £000) 16 382, −91 and 20 844 respectively, compared with the original exports from these sectors of 16 377, 4097 and 20 826. The figure of −91 indicates an import of mutton and lamb which was considered so small as to be negligible and was therefore omitted from the further calculations. The next step was to calculate the outputs, primary inputs and c_j adjustment appropriate to B and M by the procedure shown in Table 8.9 of Part 1. These values were then added to the corresponding figures derived from the known final demands to give the required results.

Results of Exercise (1)

Milk processing and cattle exports remained at approximately their original 1964 levels with mutton and lamb exports reduced to zero levels. The monetary

effects of the substitutions which are shown in Table 8.13 were as follows.

As a result of substituting home-grown for imported cereals, total imports were reduced by £7·7m. Total exports as a result of the reduction in mutton and lamb were reduced by £4·2m giving an increase in net exports of £3·5m.

Income arising in the grain growing sectors increased by £12·4m but as a result of the reduction in sheep production, income arising, in all the farming sectors combined, increased by only £8·6m. There was also a slight net reduction in non-farming industries so that the total income increase in all sectors in the model before application of the c_j adjustment was £8·2m.

The above results however are unrealistic since in the absence of subsidies they require substantial price increases throughout the system. The realistic result is obtained when the (c_j) adjustment is made which has the effect of keeping prices constant at their original 1964 levels. As can be seen, the (c_j) adjustment, which is in fact a subsidy, works out at £6·8m giving an increase (after allowing for rates and other subsidies) of £1·1m net product in all sectors in the model. If, however, we were to allow prices to rise on the home market but were to keep export prices constant, the amount of subsidy required for the latter purpose would be £1·8 million. The increase in net product in this case would be £6·2m and food prices on the home market would rise by 2·65 per cent.

Reason for these Results

It is of some interest to understand how Exercise (1) gives the original 1964 levels of B and M, with L changing. An examination of the three derived equations which are rewritten below for convenience shows that in (8.19) and (8.20) the coefficients of L are very small, so small in fact that they can be omitted, to give a system of two equations for the two unknowns B and M. When the latter are solved, the values of B and M obtained are found to be approximately at their original 1964 levels, and when these values are substituted into equation (8.21), L emerges with a very low value. Hence grain growing replaces sheep in Exercise (1).

The three equations for Exercise (1) are

$$0{\cdot}708\,799B - 0{\cdot}000\,152L - 1{\cdot}017\,360M = -9576{\cdot}2 \qquad (8.19)$$

$$0{\cdot}000\,063B + 0{\cdot}000\,151L + 0{\cdot}837\,370M = 17\,446{\cdot}1 \qquad (8.20)$$

$$0{\cdot}011\,993B + 0{\cdot}028\,146L + 0{\cdot}014\,589M = 498{\cdot}0 \qquad (8.21)$$

TABLE 8.13 Substitution of Home-grown for Imported Grains. Results of Exercises (1) and (2), compared with 1964 Original

Items	1964 Original	Exercise 1 Actual	Exercise 1 Change from Original	Exercise 2 Actual	Exercise 2 Change from Original
	£000	£000	£000	£000	£000
Imports					
Fertilisers	8 574	9 079	+505	9 071	497
Cereals	12 564	2 390	−10 174	2 390	−10 174
Other	41 514	43 513	+1 999	40 923	−591
Total Imports	62 652	54 982	−7 670	52 384	−10 268
Total Exports	137 206	133 041	−4 165	128 206	−9 000
Exports less Imports	74 554	78 059	3 505	75 822	1 268
(A) Income Arising (before the (c_j) adjustment)	207 941	216 135	8 194	214 170	6 229
of which					
(in all farming sectors)	(161 159)	(169 724)	(8 565)	(168 299)	(7 140)
(in grain-growing sectors)	(10 321)	(22 684)*	(12 363)	(22 558)†	(12 237)
(B) Adjustment (c_j)	nil	−6 844	−6 844	−6 810	−6 810
(C) Income Arising (after (c_j) adjustment)	297 941	209 291	+1 350	207 360	−581
(D) Rates	7 188	7 184	−4	7 148	−40
(E) Subsidies	−19 876	−20 097	−221	−18 718	1 158
(F) Net Product (after (c_j) adjustment) [(C) + (D) + (E)]	195 253	196 378	+1 125	195 790	537
(G) If adjustment not applied, extra subsidy or tax to keep export prices at original 1964 level	nil	−1 768	−1 768	−1 733	−1 733
(H) Net Product Arising if adjustment not applied to home market [(A) + (D) + (E) + (G)]	195 253	201 454	6 201	200 867	5 614
Price changes if no adjustment made to productive sectors			%		%
Personal Consumption	–	–	+2·65	–	+2·65
Exports	–	–	+1·33	–	+1·33

* 6350 being due to profit incentive.
† 6315 being due to profit incentive.

EXERCISE (2)
SUBSTITUTION OF HOME-GROWN FOR IMPORTED GRAIN WITH EXPORTS OF MUTTON AND LAMB HELD CONSTANT

For Exercise (2), we did precisely as for Exercise (1) except that Mutton/Lamb Exports were held at their original 1964 levels, with B and M to be determined via assumptions (1) and (3) above. This resulted in exports of Beef and Milk Products of 11 075 and 17 128 respectively compared with the original 16 377 and 20 826 (values in £000). The full results are given in Table 8.13 and show that in all cases the results are somewhat inferior to those from Exercise (1), in terms of income arising.

EXERCISE (3)
SUBSTITUTION OF IMPORTED FOR HOME-GROWN GRAIN WITH VARYING EXPORTS OF BEEF, MILK PRODUCTS, MUTTON AND LAMB

General Assumptions

To import all of the cereals home-grown in the original 1964 economy, except malting barley and native oats fed to livestock without process of sale. Replace native barley by imported (and cheaper) milo. Irish wheat is not grown; it is replaced by imported wheat. Irish oats (as specified) is replaced by imported oats. There is no restriction on imports of cereals.

Procedure

The native grains' coefficients were taken down in to the Primary Input rows and adjusted along the lines explained above for Exercise (1) and (2), except that in this case the λ's used were the reciprocals of those used in Exercises (1) and (2).

There was again a Primary row of adjustment coefficients; the new $(I - A)$ inverse was obtained and applied to the known final demands, to give part of the solution.

As in the case of Exercises (1) and (2) the following assumptions were made in order to determine the magnitudes of B, L and M, these symbols having the same meaning as in the other exercises.

(1) Cattle and Dairying outputs to be in the same proportions as for 1964 original outputs.
(2) Dairying output to be in the same ratio to Milk Processing as for the original 1964 outputs.
(3) The total land usage, as measured by the Annuity coefficient, to be unchanged.

On the basis of these assumptions, and using the same procedure as described

for Exercise (1), the following three equations were developed:

$$-0{\cdot}708\,801B + 0{\cdot}000\,151L + 1{\cdot}017\,354M = 9579{\cdot}7 \qquad (8.22)$$

$$0{\cdot}000\,290B + 0{\cdot}000\,070L + 0{\cdot}391\,488M = 8158{\cdot}1 \qquad (8.23)$$

$$0{\cdot}011\,805B + 0{\cdot}028\,058L + 0{\cdot}014\,191M = 776{\cdot}8 \qquad (8.24)$$

Solving the equations gave B and M about the same as the original 1964 amounts. L, the exports of mutton and lamb, were 10 265 compared with 1964 original of 4097 (units in £000). Thus sheep used the extra land available, in contrast to Exercise (1) where sheep production was replaced by grain growing. As in the case of Exercise (1), this result is due to the very low coefficients for L in equations (8.22) and (8.23).

The results which are given in Table 8.14 show that in this case the increase in mutton exports was not sufficient to balance the increase in grain imports so that net exports are reduced by £3·6 million. Income arising in the grain growing sectors is reduced by £8·4m. This is offset somewhat by increases in the other farming sectors but despite this, the income arising in all farming sectors is reduced by £2·8m, and in all sectors in the model by £2·2m. After the application of the c_j adjustment of £1·7m (which in this case is a surplus) and allowing for rates and subsidies, net product arising in all sectors is down by £0·1m. No attempt was made to calculate the multiplier effects, of the surplus or reduction of net product, upon the outputs of sectors, imports, etc. The results shown give the differences arising from the changes described in the exercise, when Personal Consumption, Other Industries, Stock Changes and all Exports, except those mentioned in the exercises, held constant at their original levels.

EXERCISE (4)
SUBSTITUTION OF IMPORTED FOR HOME-GROWN GRAIN WITH EXPORTS OF MUTTON AND LAMB HELD CONSTANT

For Exercise (4), we did precisely as for Exercise (3), except that Mutton/Lamb exports were held at their original 1964 levels, with B and M to be determined via equations (4) and (6) above. This resulted in exports of Beef and Milk Products of 24 352 and 26 382 respectively compared with the original 16 377 and 20 826. The full results are given in Table 8.14 and show that in all cases the results are better for the economy than those of Exercise (3), e.g. a loss of Foreign Exchange of £0·3m, compared with a corresponding loss of £3·6m for Exercise (3) and an increase of £0·7m in Income Arising as against the Exercise (3) reduction of £2·2m.

One point should be kept in mind, however, when interpreting the figures from Exercise (4), which show fairly substantial increases in beef and milk

TABLE 8.14 Substitution of Imported for Home-grown grains. Results of Exercises (3) and (4) Compared with 1964 Original

Items	1964 Original	Exercise (3) Actual	Exercise (3) Change from Original	Exercise (4) Actual	Exercise (4) Change from Original
Imports	£000	£000	£000	£000	£000
Fertilisers	8 574	7 878	−696	7 874	−700
Cereals	12 564	24 690	12 126	24 948	12 384
Other	41 514	43 142	1 628	44 760	3 246
Total Imports	62 652	75 710	13 058	77 582	14 930
Total Exports	137 206	146 675	9 469	151 822	14 616
Exports less Imports	74 554	70 965	−3 589	74 240	−314
(A) Income arising (before the (c_j) adjustment) of which	207 941	205 740	−2 201	208 592	651
(in all farming sectors)	(161 159)	(158 386)	(−2 773)	(160 274)	(−885)
(in grain growing sectors)	(10 321)	(1 922)	(−8 399)	(1 915)	(−8 406)
(B) Adjustment (c_j)	nil	+1 679	+1 679	+1 694	+1 694
(C) Income arising (after (c_j) adjustment)	207 941	207 419	−522	210 286	2 345
(D) Rates	7 188	7 194	6	7 247	59
(E) Subsidies	−19 876	−19 466	410	−21 528	1 652
(F) Net Product (after (c_j) adjustment) (C) + (D) + (E)	195 253	195 147	−106	196 005	752
(G) If adjustment not applied, extra subsidy or tax to keep export prices at original 1964 levels	nil	333	333	349	349
(H) Net product if adjustment not applied to home market (A) + (D) + (E) + (G)	195 253	193 801	−1 452	194 660	593
Price changes if no adjustment made to productive sectors:			%		%
Personal Consumption	–	–	−0·6	–	−0·6
Exports	–	–	−0·2	–	−0·2

product exports. The model assumes that these extra products could be exported for a price reduction of 0·2 per cent. In view of experience in subsequent years this is not a realistic assumption, particularly for milk products. An increase in milk product exports of this magnitude would no doubt lead to a sharp drop in prices, so that the export subsidies required would be much greater than those contained in the overall £21·5m shown here. It is doubtful therefore if the results of Exercise (4) are realistic under present world market conditions.*

SUMMARY AND CONCLUSIONS

The results of the above exercises would seem to indicate that for the economy as a whole arguments in favour of substituting cheap imported for home-grown grain are not as well founded as is often thought. If such substitution were to take place there would be either a fall, or an insignificant rise in income arising in agriculture, while the gain to the remainder of the economy from lower prices would likewise be marginal.

If on the other hand home-grown grain were substituted for imported, without any price changes, there would be a gain of £1·1m in the net product of the sectors considered in the model.† However, if, in 1964, prices were allowed to rise on the home market, but prices on export markets were kept at original levels by the application of a subsidy, the equivalent gain would be about £6m. This situation would of course involve fairly substantial income redistribution as between different classes of farmers and between farmers and other workers. Incomes of grain growers would increase by about £12m but those of other farmers would be reduced by £4–5m giving a net increase in Agricultural incomes of about £7–8m. Food prices on the home market would increase by about 2·7 per cent and direct farm subsidies by about £1–2m. As the major part of the last two expenses would be paid by the non-farming sectors, these increases would represent an income redistribution as between farmers and non-farm workers.

Unfortunately, the 1964 input–output table cannot be used to show how the substitution of home-grown grain for imports, if carried out, would affect incomes of farmers in different regions. There is no doubt that such policies would benefit farmers in the grain-growing regions of the country and work to the detriment of those on the poorer soil in the North and West, but in order to quantify the magnitudes of the benefits or penalties, regional input–output tables would be needed.

* This exercise was undertaken before Ireland decided to joint the EEC. Conditions will no doubt be different when the country becomes a full member of the European Community.

† The model under review is a partial model and does not cover all sectors of the economy. This should be kept in mind in interpreting the results.

9 LINEAR PROGRAMMING APPLIED TO INPUT–OUTPUT MODELS

Introduction

This chapter gives three numerical examples of the use of Linear Programming (L.P.) techniques, taking the national input–output structures, such as that of Table 2.3 above, as the basis of the experiments. A list of references to L.P. texts follows and some exercises for the student. These exercises use data appearing in this or earlier chapters and do not require any further L.P. computer runs as such. A technical appendix describing at some length the ideas and methods of L.P. is included at the end of this chapter.

The numerical L.P. problems which are given below are intended to be discussed sufficiently in the chapter to enable the reader to follow them through the stages of preparation for the computer and analysis of the results. A more advanced approach to these numerical problems would use reduced numbers of rows and columns for tables such as Table 9.3. The emphasis here is on explaining the methods rather than on compactness of design. The appendix is well worth study for a deeper understanding of the L.P. process and of how the solution to a problem is reached and recognized.

Differences Between Optimal and Non-Optimal Problems

The Input–Output problems encountered in the text and exercises of previous chapters have all been sets of linear equations having the number of variables to be solved for equal to the number of equations. Thus each such set of equations gave a single unique set of solutions, the typical set being the sector outputs required to satisfy specified Final Demand.

If however, as often happens, the number of variables requiring solution exceeds the number of equations, then it is necessary to assign pre-determined values to some of the variables, and subsequently solve the system of equations for the remaining variables. The allocation of values is a matter of choice as to which variables should be given a pre-determined value, usually zero. If, in addition to the set of equations, there is an algebraic expression of the variables, referred to as the Preference Function, or the Objective Function, this function to be given an optimum value, maximum or minimum, by choice of values for the variables, then the problem is referred to as a Programming Problem.

The two main differences, therefore, between an optimal or programming

problem and a non-optimal one is that the optimal problem has an expression in the variables to be maximized or minimized and that this optimization requires that certain of the variables be given pre-determined values and the remaining variables (equal to the number of equations) be then solved for. The problem is denoted a *Linear* Programming problem if the objective function and all the equations, denoted Constraints, are linear functions of the variables, i.e. have $a_{ij}x_j$ as the typical term, the a_{ij} being coefficients, such as input–output (I–A) coefficients, and x_j being the typical variable.

L.P. EXPERIMENT WITH TABLE 2.3 (U.K. 1963) REPRODUCED HERE FOR CONVENIENCE AS TABLE 9.1

The problem can be stated as follows. Maximize Consumption for the U.K. in 1963 subject to the following constraints:

(1) The import surplus must not exceed that appearing in Table 9.1. This import surplus given by total imports less total exports is 5 946—5 815 which is 131 £ million.

(2) The value of capital formation is to be kept at the same percentage of GNP as in Table 9.1 (i.e. 5 135/30 313 = 16·94 per cent). It should be noted that the figure of 30 313 is got by deducting imports (5 946) from total final demand (36 259).

(3) By recognizing limits on productive capacity, outputs of Sectors (1) to (3) in Table 9.1 are not allowed to increase more than 10 per cent above their 1963 levels. These upper limits are 2 138·4, 33 246·4, and 20 088·2 respectively. Lower limits on outputs could also be specified but we do not do so in this exercise.

In order to give the necessary flexibility for maximization to occur, Consumption and Capital Formation must be entered in the statement of the problem as variables. This is done by calculating technical coefficients for the entries in these columns in the usual way (i.e. by dividing every entry in each column by the column total). We also take the export entries in *Rows* (1) to (3) of Table 9.1 as variables, but we treat these in a slightly different way. By calculating technical coefficients we postulate fixed proportions between all entries in a column. As we do not want to impose an assumption of fixed proportions between exports of the three productive sectors we do not calculate technical coefficients for these items. We enter them as three separate variables E_1, E_2, and E_3, each having a coefficient of unity.

The technical coefficients derived from Columns (1), (2), (3), (5) and (6) of Table 9.1, as well as the variable E's are given in Table 9.2. As can be seen, the first three columns of this table are a repetition of Table 2.4, while the remainder are new columns not given previously.

TABLE 9.1 Highly Aggregated Input–Output Table for the United Kingdom, 1963

£m

Outputs →	Inter-Industry				Final Demand				
	Agric-ulture	Industry	Services	Total Inter-Industry	Con-sumption	Capital Form-ation	Exports	Total Final Demand	Output
Inputs ↓	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
1. Agriculture	277	444	14	735	1 123	35	51	1 209	1 944
2. Industry	587	11 148	1 884	13 619	8 174	4 497	3 934	16 605	30 224
3. Services	236	2 915	1 572	4 723	11 657	430	1 452	13 539	18 262
4. Total Inter-Industry	1 100	14 507	3 470	19 077	20 954	4 962	5 437	31 353	50 430
Primary Inputs									
5. Imports	133	2 844	676	3 653	1 770	250	273	2 293	5 946
6. Sales by Final Buyers	3	134	42	179	–90	–177	88	–179	-
7. Indirect Taxes less Subsidies	–246	499	442	695	2 675	100	17	2 792	3 487
8. Wages, Salaries, Profits	954	12 240	13 632	26 826	-	-	-	-	26 826
9. Total Primary Inputs	844	15 717	14 792	31 353	4 355	173	378	4 906	36 259
10. Total Inputs	1 944	30 224	18 262	50 430	25 309	5 135	5 815	36 259	86 689

TABLE 9.2 Enlarged Table of Technical Coefficients for United Kingdom 1963

Inputs ↓	Agriculture (1)	Industry (2)	Services (3)	Consumption (4)	Capital Formation (5)	Exports		
						Agri. (6)	Indust. (7)	Services (8)
1. Agriculture	0·1425	0·0147	0·0008	0·0444	0·0068	1·0		
2. Industry	0·3020	0·3688	0·1032	0·3230	0·8758		1·0	
3. Services	0·1214	0·0964	0·0861	0·4606	0·0837			1·0
4. Imports	0·0684	0·0941	0·0370	0·0699	0·0487			
5. Sales by F.B.	0·0015	0·0044	0·0023	−0·0036	−0·0345			
6. Indirect Taxes less Subsidies	−0·1265	0·0165	0·0242	0·1057	0·0195			
7. Wages etc.	0·4907	0·4050	0·7465	-	-			
8. Total Inputs	1·0	1·0	1·0	1·0	1·0	1·0	1·0	1·0

Formulating the Programme

Let X_1, X_2 and X_3 be the symbols for the outputs of agriculture, industry, and services respectively. Let C be consumption, CF capital formation and E_1, E_2 and E_3 be the exports of agricultural goods, industrial goods, and services respectively. Then, using the data in the first three rows of Table 9.2 we can state a set of 3 equations as follows:

$$\begin{aligned}
(1)\, X_1 &= 0{\cdot}1425X_1 + 0{\cdot}0147X_2 + 0{\cdot}0008X_3 + 0{\cdot}0444C + 0{\cdot}0068CF + 1{\cdot}0E_1\\
(2)\, X_2 &= 0{\cdot}3020X_1 + 0{\cdot}3688X_2 + 0{\cdot}1032X_3 + 0{\cdot}3230C + 0{\cdot}8758CF + 1{\cdot}0E_2\\
(3)\, X_3 &= 0{\cdot}1214X_1 + 0{\cdot}0964X_2 + 0{\cdot}0861X_3 + 0{\cdot}4606C + 0{\cdot}0837CF + 1{\cdot}0E_3
\end{aligned} \tag{9.1}$$

We next state a set of 4 equations for imports (IM), sales to final buyers (FB), indirect taxes less subsidies (T–S), and wages, salaries, and profits (WSP), using the relevant data for these items from Table 9.2 and the actual exports of these items given in Column 7 of Table 9.1. The equations are as follows:

$$\begin{aligned}
(4)\, IM &= 0{\cdot}0684X_1 + 0{\cdot}0941X_2 + 0{\cdot}0370X_3 + 0{\cdot}0699C + 0{\cdot}0487CF + 273{\cdot}0\\
(5)\, FB &= 0{\cdot}0015X_1 + 0{\cdot}0044X_2 + 0{\cdot}0023X_3 - 0{\cdot}0036C - 0{\cdot}0345CF + 88{\cdot}0\\
(6)\, T\text{-}S &= -0{\cdot}1265X_1 + 0{\cdot}0165X_2 + 0{\cdot}0242X_3 + 0{\cdot}1057C + 0{\cdot}0195CF + 17{\cdot}0\\
(7)\, WSP &= 0{\cdot}4907X_1 + 0{\cdot}4050X_2 + 0{\cdot}7465X_3 + 0{\cdot}0C + 0{\cdot}0CF + 0{\cdot}0
\end{aligned} \tag{9.2}$$

Now in order to use all the above 7 equations in the linear programming model they must be put in the $(\mathrm{I} - A)$ format, which means changing the sign and adding unity in the diagonal position. Thus equation (1) of (9.1), when rearranged, becomes

$$0{\cdot}8575X_1 - 0{\cdot}0147X_2 - 0{\cdot}0008X_3 - 0{\cdot}0444C - 0{\cdot}0068CF - 1{\cdot}0\mathrm{E}_1 = 0{\cdot}0 \tag{9.3}$$

and equation (4) of (9.2) becomes

$$-0{\cdot}0684X_1 - 0{\cdot}0941X_2 - 0{\cdot}0370X_3 - 0{\cdot}0699C - 0{\cdot}0487CF + IM = 273{\cdot}0 \tag{9.4}$$

The other 5 equations are rearranged in a similar manner and all are entered in the rearranged form in the first 7 rows of Table 9.3.

The next operation is to state a set of equations for the other constraints which we outlined at the start of this section. To do this we state first (where necessary) a set of inequalitites, and later, by adding what we call slack variables, turn the inequalities into equations. Let us take each of these constraints in turn.

The first constraint is that the import surplus (which is given by total imports less total exports) must not exceed £131 million. Now since total imports is represented by the variable IM, and total exports is made up of the three

TABLE 9.3 $(I-A)$ Format for Linear Programming Problem Based on Table 2.3

Constraint Row	Activities					
	X_1	X_2	X_3	C	CF	E_1
	(1)	(2)	(3)	(4)	(5)	(6)
(1) Agriculture	0·8575	—0·0147	—0·0008	—0·0444	—0·0068	—1·0
(2) Industry	—0·3020	0·6312	—0·1032	—0·3230	—0·8758	
(3) Services	—0·1214	—0·0964	0·9139	—0·4606	—0·0837	
(4) Imports	—0·0684	—0·0941	—0·0370	—0·0699	—0·0487	
(5) Final Buyers	—0·0015	—0·0044	—0·0023	0·0036	0·0345	
(6) Indirect Taxes less Subsidies	0·1265	—0·0165	—0·0242	—0·1057	—0·0195	
(7) Wages, Salaries, Profits	—0·4907	—0·4051	—0·7464			
(8) Import Excess						—1·0
(9) Capital Formation				—0·1694	0·8306	—0·1694
(10) Upper Limit, Agriculture	1·0					
(11) Upper Limit, Industry		1·0				
(12) Upper Limit, Services			1·0			
Coefficients of Objective Function				1·0		

Activities										Original Form of Inequality	Constant £ million
E_2	E_3	IM	FB	$T-S$	WSP	S_8	S_{10}	S_{11}	S_{12}		
(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)		
										=	0·0
−1·0										=	0·0
	−1·0									=	0·0
		1·0								=	273·0
			1·0							=	88·0
				1·0						=	17·0
					1·0					⩽	0·0
−1·0	−1·0	1·0				1·0				⩽	509·0
−0·1694	−0·1694	0·1694								=	64·0
							1·0			⩽	2 138·4
								1·0		⩽	33 246·4
									1·0	⩽	20 088·2
										MAX.	

variables $E_1 + E_2 + E_3$ plus the aggregate of the exports in rows 5 to 8 of column 7 of Table 9.1 (i.e. 378), we can write this constraint in the form of the following inequality

$$IM - (E_1 + E_2 + E_3 + 378) \leqslant 131 \qquad (9.5)$$

where the sign $\leqslant$ means equal to or less than. Rearranging this inequality so as to bring the -378 across to the right-hand side (where its sign changes to plus) we obtain

$$1{\cdot}0IM - 1{\cdot}0E_1 - 1{\cdot}0E_2 - 1{\cdot}0E_3 \leqslant 509 \qquad (9.6)$$

and turning this inequality into an equation by the addition of a slack variable designated S_8 we obtain

$$IM - 1{\cdot}0E_1 - 1{\cdot}0E_2 - 1{\cdot}0E_3 + S_8 = 509 \qquad (9.7)$$

This equation is entered without further rearrangements as Row 8 of Table 9.3.

The second constraint is that the value of capital formation is to be kept at 16·94 per cent of GNP. If we define GNP as consumption (C) plus capital formation (CF) plus variable exports ($E_1 + E_2 + E_3$) plus constant exports (378 units), minus imports (IM), then the capital formation (CF) constraint equation is

$$CF = 0{\cdot}1694(C + CF + E_1 + E_2 + E_3 + 378 - IM) \qquad (9.8)$$

Rearrangement of (9.8) gives

$$-0{\cdot}1694C + 0{\cdot}8306CF - 0{\cdot}1694E_1 - 0{\cdot}1694E_2 - 0{\cdot}1694E_3 + 1{\cdot}0IM = 64 \qquad (9.9)$$

This equation is entered in Row 9 of Table 9.3. The third constraint is that outputs of sectors (1) to (3) in Table 9.1 are not to increase above 2 138·4, 33 246·4, and 20 088·2 units respectively. This constraint requires three equations which are derived as follows:

$$\begin{aligned} X_1 &\leqslant 2\,138{\cdot}4 \\ X_2 &\leqslant 33\,246{\cdot}4 \\ X_3 &\leqslant 20\,088{\cdot}2 \end{aligned} \qquad (9.10)$$

Changing these inequalitites into equations by the addition of the slack variables S_{10}, S_{11} and S_{12}, we obtain the following three equations

$$\begin{aligned} 1{\cdot}0X_1 + S_{10} &= 2\,138{\cdot}4 \\ 1{\cdot}0X_2 + S_{11} &= 33\,246{\cdot}4 \\ 1{\cdot}0X_3 + S_{12} &= 20\,088{\cdot}2 \end{aligned} \qquad (9.11)$$

These are entered in Rows (10) (11) and (12) of Table 9.3.

A linear expression for the objective function is required. Since our objective

is to maximize consumption (C) without any modification, we say that

$$1{\cdot}0C = \text{maximum}$$

This expression is entered in the last row of Table 9.3. The problem is now ready for computer processing. An explanation of the method of processing is outside the scope of this book, but if a table such as 9.3 is given to a computer programmer he can have it processed very easily and the result obtained in a short time.

TABLE 9.4 Results of L.P. Experiment Compared with Entries in Table 9.1

Sector/Activity/GNP Component	Table 9.1 1963 Actual Levels (1)	L.P. Optimal Levels (2)	Difference: (2) less (1) (3)
	£ million	£ million	£ million
(1) Agriculture	1 944	2 138·40	194·40
(2) Industry	30 224	33 246·40	3 022·40
(3) Services	18 262	17 901·20	−360·80
(4) Imports	5 946	6 307·29	361·29
(5) Sales by Final Buyers	0·0	0·00	0·00
(6) Indirect Taxes less Subsidies	3 487	3 610·75	123·75
(7) Wages, Salaries, Profits	26 826	27 878·89	1 052·89
(8) Consumption	25 309	26 286·33	977·33
(9) Capital Formation	5 135	5 334·31	199·31
(10) Agricultural Exports	51	127·25	76·25
(11) Industrial Exports	3 934	5 329·65	1 395·65
(12) Services' Exports	1 452	341·39	−1 110·61
(10) + (11) + (12)	5 437	5 798·29	361·29
(13) Total Exports	5 815	6 176·29	361·29
(14) Import Excess	131	131·00	0·00
(15) GNP Arising (6) + (7)	30 313	31 489·64	1 176·64
(16) Expenditure on the GNP (8) + (9) − (14)	30 313	31 489·64	1 176·64

Results of Experiment Based on Table 9.1

The results of the experiment as taken from the computer output are given in Table 9.4, together with the 1963 actual levels taken from Table 9.1. As can be seen from Table 9.4 the optimum level of GNP arising is £1 177 million above the actual 1963 level. This comes about as a result of the various changes shown in Column (3) of the Table. Some of the more important of these changes are increases of £3 022 million in industrial output, £1 396 million in industrial exports, £1 053 million in wages, salaries and profits, and a decline of £1 111 million in services' exports.

It is not suggested that these results are to be taken seriously. To obtain realistic results in an experiment of this nature would require a highly disaggregated I—O table and the specification of a large number of constraints. For example, the optimum solution given in Table 9.4 shows an increase in industrial exports of about one-third, which is far more optimistic than could be hoped for in any short-run situation. It would seem realistic therefore in a practical situation to place upper bounds on industrial exports. Similarly the decline of about 80 per cent in services' exports is hardly acceptable and it seems that in a practical situation lower bounds would need to be placed on these.

Unless the planner is careful not to place too rigid constraints on the programme, he pre-determines his results to a great extent and learns little from the exercise. What should be done in practice is to specify fairly wide ranges, within which sectors can vary, and to observe the directions in which any changes occur. This observation indicates how the economy should move in order to maximize the objective function and gives planners an idea of the types of policies which should be initiated.

L.P. EXPERIMENT WITH TABLE 2.1 (IRELAND 1960)

In the previous experiment we showed how to formulate a linear programming problem where the objective was to maximize some function, i.e. consumption. In this exercise we show how to set up a programme where the objective is to minimize a function, and choose for this purpose the minimization of imports for the same level of GNP as in Table 2.1, i.e. £669·080 million.

The constraints specified for this programme are:

(a) Each item of government consumption and capital formation is kept at the same level as in Table 2.1.

(b) Household consumption is allowed to vary, using the input pattern given in Column (5) of Table 2.1, the input coefficients for this pattern being obtained by dividing each entry in Column (5) by the total input of 502·571.

(c) As in the previous experiment, exports of the three productive sectors in Column (8) are allowed to vary but the remaining entries in Column (8) are held constant.

(d) Outputs of the three productive sectors (i.e. agriculture, industry, and services) are not to increase by more than 10 per cent.

Setting up the Equations

The technical coefficients and other data required for formulating the programme are given in Table 9.5.

By use of the data in this table the set of equations given in Table 9.6 is stated. The first equation is

$$X_1 = 0{\cdot}0109X_1 + 0{\cdot}1518X_2 + 0{\cdot}0038X_3 + 0{\cdot}1236C + E_1 + 3{\cdot}474 \tag{9.12}$$

Bringing all the variables to the left-hand side gives the following equation, in the $(I - A)$ format, which is entered as Row 1 of Table 9.6.

$$0{\cdot}9891X_1 - 0{\cdot}1518X_2 - 0{\cdot}0038X_3 - 0{\cdot}1236C - 1{\cdot}0E_1 = 3{\cdot}474 \tag{9.13}$$

The next two rows of Table 9.6 are stated in a similar manner. The fourth row is stated as follows:

$$IM = 0{\cdot}0763X_1 + 0{\cdot}2227X_2 + 0{\cdot}0261X_3 + 0{\cdot}1240C + 31{\cdot}092 \tag{9.14}$$

and when written in the $(I - A)$ format this equation becomes

$$1{\cdot}0IM - 0{\cdot}0763X_1 - 0{\cdot}2227X_2 - 0{\cdot}0261X_3 - 0{\cdot}1240C = 31{\cdot}092 \tag{9.15}$$

which is the equation entered in line 4 of Table 9.6. Rows 5-8 are stated in a similar manner to Row 4.

In order to state the equation for GNP we first define this item as indirect taxes (T), less subsidies (S), plus wages, salaries and profits (WSP), plus depreciation (D). Using this definition the GNP equation becomes $1{\cdot}0T - 1{\cdot}0S + 1{\cdot}0WSP + 1{\cdot}0D = 669{\cdot}080$, and is entered in this form in Row 9 of Table 9.6.

The final three rows of Table 9.6 are derived from the upper limits specified for Agriculture, Industry, and Services thus

$$\begin{aligned} 1{\cdot}0X_1 &\leqslant 220{\cdot}379 \\ 1{\cdot}0X_2 &\leqslant 591{\cdot}931 \\ 1{\cdot}0X_3 &\leqslant 331{\cdot}442 \end{aligned} \tag{9.16}$$

Turning these three inequalitites into equations by the addition of slack variables S_{10}, S_{11} and S_{12} we obtain

$$\begin{aligned} 1{\cdot}0X_1 + S_{10} &= 220{\cdot}379 \\ 1{\cdot}0X_2 + S_{11} &= 591{\cdot}931 \\ 1{\cdot}0X_3 + S_{12} &= 331{\cdot}442 \end{aligned} \tag{9.17}$$

The final entry in Table 9.6 is the objective function. The objective is to minimize imports, but in linear programming terminology this objective is redefined to become the maximization of negative imports and is written at the bottom

TABLE 9.5 Technical Coefficients and Other Data Required for Preparing the L.P. Experiment Based on Table 2.1

	Agriculture	Industry	Services	Household Consumption	Exports	Government Consumption	Capital Formation	Total Constraints
	X_1	X_2	X_3	C				
		Technical Coefficients				Variables and Constants (£m)		
(1) Agriculture (X_1)	0·0109	0·1518	0·0038	0·1236	E_1	0·803	2·671	3·474
(2) Industry (X_2)	0·1383	0·1822	0·0845	0·4120	E_2	14·821	61·732	76·553
(3) Services (X_3)	0·0550	0·0599	0·0647	0·2770	E_3	50·847	6·428	57·277
(4) Imports (*IM*)	0·0763	0·2227	0·0261	0·1240	3·345	1·764	25·983	31·092
(5) Indirect Taxes (*T*)	0·0577	0·0915	0·0305	0·0634	3·175	-	0·886	4·061
(6) Subsidies (*S*)	0·0365	0·0109	0·0222	-	-	-	-	-
(7) Wages etc. (*WSP*)	0·6669	0·2796	0·7618	-	33·912	-	-	33·912
(8) Depreciation (*D*)	0·0314	0·0232	0·0508	-	-	2·500	−1·100	1·400
(9) GNP	-	-	-	-	-	-	-	669·080
Upper Limits £ million	220·379	591·931	331·442	-	-	-	-	-

of Table 9.6 as

$$-1{\cdot}0IM = \text{maximum}$$

The programme is now ready for computer processing.

Results of Experiment Based on Table 2.1

The optimum levels of output, wages, salaries etc. as taken from the computer results are given in Column (2) of Table 9.7, together with the 1960 actual levels in Column (1). Additional results are given in Column (3) and are explained below.

The most outstanding result in Column (2) is that household consumption is zero. This is of course a completely unrealistic situation and if such a result were to be obtained in an actual planning situation the problem should be reformulated putting a lower bound (B) on household consumption. This could be done by the inclusion of a further equation in Table 9.6. Such an equation could be derived from the inequality

$$1{\cdot}0HC \geqslant B, \text{ rewritten} -1{\cdot}0HC \leqslant -B,$$

by the addition of a slack variable, to obtain

$$-1{\cdot}0HC + S_i = -B$$

where HC = household consumption; the sign $\geqslant$ means greater than or equal to, and B is the lower bound which might be the 1960 level of £502·571 million. Any other level of consumption could of course be selected.

Other interesting features of the results are that agriculture and services are operating at their permitted upper limits with industry reduced below its actual 1960 level. These results indicate that if the objective is to minimize imports then we should expand both agricultural production and services (items having low import contents) and reduce industry, which has a high import content. Table 9.7 shows that when this is done subsidies (which go mainly to agriculture) are increased, and indirect taxes, which are paid on many items of household consumption, are reduced. Wages, salaries, and profits are also reduced. The latter is a most unwelcome result indicating that a policy of reducing imports would have the undesirable side effect of increasing unemployment. Hence a policy of increasing imports might be much more acceptable, particularly if exports were expanded accordingly. The effects of such a policy could easily be tested by a rearrangement of the data in Table 9.6.

Shadow Prices

Column (3) of Table 9.7 gives what are called the *shadow prices* (S.P.'s). It will be noticed that there is such a price for each constraint row of Table 9.6

TABLE 9.6 $(I-A)$ Format for L.P. Problem Based on Table 2.1

Constraint Row	Activities						
	X_1	X_2	X_3	*IM*	*T*	*S*	*WSP*
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
(1) Agriculture	0·9891	—0·1518	—0·0038				
(2) Industry	—0·1383	0·8178	—0·0845				
(3) Services	—0·0550	—0·0599	0·9353				
(4) Imports	—0·0763	—0·2227	—0·0261	1·0			
(5) Indirect Taxes	—0·0577	—0·0915	—0·0305		1·0		
(6) Subsidies	—0·0365	—0·0109	—0·0222			1·0	
(7) Wages, Salaries, Profits	—0·6669	—0·2796	—0·7618				1·0
(8) Depreciation	—0·0314	—0·0232	—0·0508				
(9) GNP					1·0	—1·0	1·0
(10) Upper Limit, Agriculture	1·0						
(11) Upper Limit, Industry		1·0					
(12) Upper Limit, Services			1·0				
Coefficients of Objective Function				—1·0			

Activities								Original Form of Inequality	Constants £ million
D	C	E_1	E_2	E_3	S_{10}	S_{11}	S_{12}		
(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)		
	−0·1236	−1·0						=	3·474
	−0·4120		−1·0					=	76·553
	−0·2770			−1·0				=	57·277
	−0·1240							=	31·092
	−0·0634							=	4·061
								=	0·0
								=	33·912
1·0								=	1·400
1·0								=	669·080
					1·0			≤	220·379
						1·0		≤	591·931
							1·0	≤	331·442
								MAX.	

(12 in all). However, the first three of these (i.e. those for Agriculture, Industry, and Services), all of which are zero, are of little significance. The significant S.P's for these activities are those given in Rows 10, 11 and 12 of Column (3) (i.e. those for the upper limits on Agriculture, Industry, and Services). Indeed in all cases where there is an equation for an upper or lower bound on an activity, the shadow price of importance for that activity is the S.P. of the bound equation and not that of the activity equation.

TABLE 9.7 Results of the L.P. Experiment Based on Table 2.1

£ million

Sector/Activity/Constraint/ GNP Component	1960 Actual Levels (1)	L.P. Optimal Levels (2)	Shadow Prices of Constraint Rows for Optimal (3)
(1) Agriculture	200·345	220·379	0·0
(2) Industry	538·119	519·206	0·0
(3) Services	301·311	331·442	0·0
(4) Imports	236·378	172·185	− 1·0
(5) Indirect Tax	105·961	74·393	+ 0·5809
(6) Subsidies	19·866	21·061	− 0·5809
(7) Wages/Salaries/Profits	547·485	478·545	+ 0·5809
(8) Depreciation	35·500	37·203	+ 0·5809
(9) Gross National Product Arising [(5) − (6) + (7) + (8)]	669·080	669·080	− 0·5809
(10) Agriculture Upper Limit		220·379	+ 0·3416
(11) Industry Upper Limit		591·931	0·0
(12) Services Upper Limit		331·442	+ 0·4507
(13) Household Consumption	502·571	ZERO	
(14) Government Consumption	70·737	70·737	
(15) Capital Formation	96·600	96·600	
(16) Agricultural Exports	49·750	134·428	
(17) Industrial Exports	103·278	289·569	
(18) Service Exports	42·090	209·499	
[(16) + (17) + (18)]	195·118	633·496	
(19) Total Exports	235·550	673·928	
(20) Import Excess [(4) − (19)]	0·828	−501·743	
(21) Expend. on the GNP [(13) + (14) + (15) − (20)]	669·080	669·080	

The meaning of the shadow prices is explained in the Appendix to this chapter. It suffices to say here that the shadow price of a constraint is the marginal product of that constraint at the level at which it is included in the solution. This is the amount which would be gained or sacrificed (in units of the objective function) from using or leaving unused one extra unit of each constraint constant. Thus since the shadow price of imports is — 1·0 this means that the objective function (minimization of imports) would be reduced by one unit if an extra unit of imports were used. Similarly imports would be increased by £0·5809 if GNP were increased by an extra £1. On the other hand imports would be reduced by an extra £0·3416 if an extra unit of agriculture were to be produced. Since, however, the upper bound for industry is not reached, the limit prescribed neither adds to nor takes from the objective function, as indicated by the shadow price of zero opposite "Industry Upper Limit". This always happens in such cases. Where a bound is not reached the shadow price of the constraint in question is zero.

Since shadow prices tell us the gain or loss from the inclusion of an extra unit of a constraint they are of great importance to the planner. They tell him by how much the objective function is affected, per unit of each of the bounds which he has placed on different activities and they also indicate the constraints which are best relaxed in order to increase the objective function further. For example in the present exercise an increase in the bounds placed on Agriculture and Services would reduce Imports below the level shown in Column (2) of Table 9.7, whereas an increase in the bound on Industry would have no effect on Imports, since Industry would not reach that bound unless forced to do so by some further constraint.

Properties of Shadow Prices

Shadow prices have two important properties which can be used for various purposes but particularly to check that the arithmetic of the calculations is correct. The computer will not of course make arithmetical errors but if the input cards are wrongly punched an incorrect answer will be obtained. It is useful therefore to be able to check the computer results for incorrect inputs, via this property.

The check is to multiply the individual entries in the column of constants in the L.P. tableau (i.e. Table 9.6) by the shadow prices for their respective rows, and sum the products. If the arithmetic is correct this summation gives the optimum for the objective function, as formulated. Thus if we multiply the column of constants in Table 9.6 individually by the corresponding shadow prices in Table 9.7, and sum the results, we should obtain (except for rounding errors) the L.P. optimal level of imports in Column (2) of Table 9.7, which with negative sign is — 172·185. This latter negative value is in fact the negative maximum, i.e. the minimum, as formulated in the last row of Table 9.6.

The calculations are as follows:

Constant £ million		Shadow Prices*		Result
3·474	x	0·0	=	0·0
76·553	x	0·0	=	0·0
57·277	x	0·0	=	0·0
31·092	x	− 1·0	=	− 31·09200
4·061	x	+ 0·58086	=	+ 2·35887
0·0	x	− 0·58086	=	0·0
33·912	x	+ 0·58086	=	+ 19·69812
1·400	x	+ 0·58086	=	+ 0·81320
669·080	x	− 0·58086	=	− 388·64181
220·379	x	+ 0·34163	=	+ 75·28808
591·931	x	0·0	=	0·0
331·442	x	+ 0·45072	=	+ 149·38754
				− 172·18800

* To minimize rounding errors we have taken the shadow prices to 5 decimal places compared with 4 places in Table 9.7.

The second property of the shadow prices is as follows. If the coefficients of any of the twelve Table 9.6 columns selected by the optimal set of activities are multiplied by the shadow prices for their respective rows, and the products summed, the result is equal to the coefficient of the objective function for that column. Readers should verify that the entries in the first column of Table 9.6, when multiplied by their respective shadow prices, sum to zero, which is the coefficient of the objective function for that column, as shown in the last row of Table 9.6.

L.P. EXPERIMENT WITH TABLE 2.7 (NETHERLANDS 1956)

Problem

If the exports of some one sector of the Netherlands economy are to be increased by 1 million guilders, decide by an L.P. experiment which sector should be selected so as to maximize household consumption, and show the effects of this policy on the other sectors of the economy.

The following constraints apply:

(1) Extra exports are to be exactly balanced by imports, leaving zero trade surplus or deficit for the increased economic activity.

(2) The aggregate value of (a) government consumption, (b) gross domestic

TABLE 9.8 Technical Coefficients and Other Data Required for Preparing the L.P. Experiment based on Table 2.7

	Agric. X_1	Metals X_2	Textiles X_3	Mining etc. X_4	Trade X_5	Services X_6	Exports E_i	T.M. Exports	S.M. Exports	House Cons. HC	RFD
(1) Agriculture etc.	0·3993	-	0·0088	0·0163	0·0001	0·0298	$1{\cdot}0E_1$	0·15	0·15	0·2855	—0·0013
(2) Metals etc.	0·0263	0·2226	0·0140	0·0373	0·0217	0·0726	$1{\cdot}0E_2$	0·15	0·15	0·0431	0·4758
(3) Textiles etc.	0·0016	0·0025	0·2665	0·0051	0·0028	0·0033	$1{\cdot}0E_3$	0·15	0·15	0·1096	0·0092
(4) Mining etc.	0·0435	0·0929	0·0729	0·2064	0·0596	0·0597	$1{\cdot}0E_4$	0·15	0·15	0·0881	0·0369
(5) Trade	0·0251	0·0442	0·0297	0·0256	0·0088	0·0075	-	-	-	0·1962	0·0404
(6) Services	0·0167	0·0333	0·0319	0·0389	0·1650	0·0887	$1{\cdot}0E_6$	-	-	0·1912	0·0391
(7) Imports	0·1584	0·2127	0·2628	0·2817	0·0463	0·1368				0·0861	0·1754
(8) Depreciation	0·0251	0·0185	0·0215	0·0582	0·0356	0·0956				-	0·0130
(9) Net Indirect Taxes	0·0397	0·0339	0·0113	0·0459	0·1549	0·0190				-	-
(10) Employees' Income	0·0862	0·2165	0·1827	0·1834	0·1647	0·2746				-	-
(11) Profits	0·1782	0·1228	0·0978	0·1012	0·3405	0·2124				-	0·0077
(12) Exports	-									-	-
(13) RFD	-									0·7028	1·0000

fixed capital formation, and (c) net increase in stocks, is to be kept in the same proportion to household consumption as in Table 2.7. The value of (a) + (b) + (c) above as taken from Columns (10), (11) and (12) of Table 2.7, and referred to below as residual final demand (RFD), is 4 913 + 8 119 + 723 = 13 755 million guilders. The total value of Household Consumption as taken from Column 9 of Table 2.7 is 19 753 million guilders, so that the ratio between these aggregates is 13 755/19 753 = 70·28 per cent.

The technical coefficients and other data required for the solution of the problem, taken from Tables 2.7 and 2.8, are shown in Table 9.8. In preparing this table, technical coefficients have been calculated for Household Consumption (HC) and Residual Final Demand RFD.

Setting out the Equations

The equations for the solution of the problem are set out as shown in Table 9.9. As in previous experiments these equations are in the $(I-A)$ format. In Table 9.9 exports of the different sectors are denoted by E_1, E_2, E_3, E_4 and E_6. Trade exports, which should be denoted by the symbol E_5, are omitted since these exports are really the trade margins on the exports of agriculture, metals, textiles etc. If we use DEP for depreciation, NIT for net indirect taxes, EI for employees' income and PRO for profits we can state the system of equations as follows:

Equation 1 is:

$$X_1 = 0{\cdot}3993X_1 - 0{\cdot}0X_2 + 0{\cdot}0088X_3 + 0{\cdot}0163X_4 + 0{\cdot}0001X_5 + 0{\cdot}0298X_6 + 1{\cdot}0E_1 + 0{\cdot}2855\ \mathrm{HC} - 0{\cdot}0013\ \mathrm{RFD} \tag{9.18}$$

Changing this into the $(I-A)$ format gives

$$0{\cdot}6008X_1 + 0{\cdot}0X_2 - 0{\cdot}0088X_3 - 0{\cdot}0163X_4 - 0{\cdot}0001X_5 - 0{\cdot}0298X_6 - 1{\cdot}0E_1 - 0{\cdot}2855\ \mathrm{HC} + 0{\cdot}0013\ \mathrm{RFD} = 0 \tag{9.19}$$

which becomes the first equation in Table 9.9. The next three equations of Table 9.9 are prepared in exactly the same way.

The trade equation which is fifth in the table is prepared as shown below and then changed into the $(I-A)$ format

$$X_5 = 0{\cdot}0251X_1 + 0{\cdot}0442X_2 + 0{\cdot}0297X_3 + 0{\cdot}0256X_4 + 0{\cdot}0088X_5 + 0{\cdot}0075X_6 + 0{\cdot}15E_1 + 0{\cdot}15E_2 + 0{\cdot}15E_3 + 0{\cdot}15E_4 + 0{\cdot}0E_6 + 0{\cdot}1962\ \mathrm{HC} + 0{\cdot}0404\ \mathrm{RFD} \tag{9.20}$$

It should be noted that the figure of 0·15, which is the coefficient of each of the exports E_1 to E_4, is the trade and transport margin on each additional unit of export entering the system. It is assumed that there is no trade margin on E_6 which is the export of services.

The equation for services (equation 6) is stated in a similar manner on the

assumption that there is a service margin of 15 per cent on each variable export with the exception of services themselves. Equations (7) to (11) are designed in the same way as equations (1) to (4). Equation (12) for exports is entered in Table 9.9 as follows:

$$1{\cdot}3E_1 + 1{\cdot}3E_2 + 1{\cdot}3E_3 + 1{\cdot}3E_4 + 1{\cdot}0E_6 = 100 \qquad (9.21)$$

where each 1·3 coefficient is the value of a unit of exports plus the trade and service margins on it (i.e. 1·0 + 0·15 + 0·15).

Equation (13) represents the constraint stated above, that household consumption must represent 70·28 per cent of RFD i.e.

$$1{\cdot}0\text{RFD} = 0{\cdot}7028\text{HC}$$

Therefore $$1{\cdot}0\text{RFD} - 0{\cdot}7028\text{HC} = 0{\cdot}0 \qquad (9.22)$$

Finally the objective function is written as

$$1{\cdot}0\text{HC} = \text{maximum}$$

The equations are formed in such a way that only one of the exports comes into the final solution. This happens for the following reasons.

As can be seen from Table 9.9 there are 17 unknowns in the system, represented by Columns (1) - (17), but there are only 13 equations as represented by Rows (1) - (13). Therefore no more than 13 of the 17 unknown can come into the final solution, with 4 being left out.

Household consumption HC comes in because the objective is to maximize this item. RFD comes in because it is linked in equation 13 with HC, and if one comes in so must the other. In a similar manner items (1) to (6) must come in because they also are linked to HC. Imports must come in because they must equal exports and we have specified that 100 units of exports must come in. DEP, NIT, EI and PRO must also come in because these items are tied to the technical coefficients in the first six columns of Table 9.9. For example, the entries in the first six columns opposite Net Indirect Taxes (NIT) are negative, hence they must be balanced by a positive NIT entry if the row total is to be zero as specified. Similarly for the other items.

As we have now accounted for 12 items, only one further item remains to come in and this must be one of the 5 exports specified.

Results of Experiment Based on Table 2.7

The results of the experiment based on Table 2.7 are given in Table 9.10 together with the 1956 actual levels and the shadow prices. As can be seen, services is the export to be introduced, with nil values for the other 5 export items. This export generates 1·32 million guilders of household consumption and 2·24 million GNP. The multiplier effect therefore of an extra unit of service exports is 2·24 units of GNP.

TABLE 9.9 $(I - A)$ Format for L.P. Problem Based on Table 2.7

Constraint Row	Activities						
	X_1	X_2	X_3	X_4	X_5	X_6	DEP
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
(1) Ag. etc.	0·6008		—0·0088	—0·0163	—0·0001	—0·0298	
(2) Metals etc.	—0·0263	0·7773	—0·0140	—0·0373	—0·0217	—0·0726	
(3) Textiles etc.	—0·0016	—0·0025	0·7334	—0·0051	—0·0028	—0·0033	
(4) Mining etc.	—0·0435	—0·0929	—0·0729	0·7936	—0·0596	—0·0597	
(5) Trade	—0·0251	—0·0442	—0·0297	—0·0256	0·9912	—0·0075	
(6) Services	—0·0167	—0·0333	—0·0319	—0·0389	—0·1650	0·9113	
(7) Imports	—0·1584	—0·2127	—0·2628	—0·2817	—0·0463	—0·1368	
(8) Deprec.	—0·0251	—0·0185	—0·0215	—0·0582	—0·0356	—0·0956	1·0
(9) Net Indirect Tax	—0·0397	—0·0339	—0·0113	—0·0459	—0·1549	—0·0190	
(10) Employees' Income	—0·0862	—0·2165	—0·1827	—0·1834	—0·1647	—0·2746	
(11) Profits	—0·1782	—0·1228	—0·0978	—0·1012	—0·3405	—0·2124	
(12) Exports							
(13) RFD							
Objective Function							

Activities										Original Form of Inequality	Constants, in Value Units
NIT	EI	PRO	E_1	E_2	E_3	E_4	E_6	HC	RFD		
(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)		
			–1.0					–0·2855	0·0013	=	0·0
				–1·0				–0·0431	–0·4758	=	0·0
					–1·0			–0·1098	–0·0092	=	0·0
						–1·0		–0·0881	–0·0389	=	0·0
			–0·15	–0·15	–0·15	–0·15		–0·1962	–0·0404	=	0·0
			–0·15	–0·15	–0·15	–0·15	–1·0	–0·1912	–0·0391	=	0·0
								–0·0861	–0·1754	=	–100·0
									–0·0130	=	0·0
1·0										≤	0·0
	1·0								–0·2018	=	0·0
		1·0							–0·0077	=	0·0
			1·3	1·3	1·3	1·3	1·0			=	100·0
								–0·7028	1·0	=	0·0
								1·0		MAX.	

TABLE 9.10 Results of the L.P. Experiment Compared with Entries in Table 2.7

Sector/Activity/Constraint/ GNP Component	Table 2.7 1956 Actual Levels (1)	L.P. Optimal Levels (2)	Constraint Rows' Shadow Prices Optimal (3)
	million guilders		
(1) Agriculture, Fish, Food	16 076	0·718	−0·5357
(2) Metals & Construction	15 481	0·848	−0·5677
(3) Textiles & Apparel	4 417	0·225	−0·7087
(4) Mining, Chem., Util.	10 709	0·497	−0·6625
(5) Trade	7 521	0·386	−0·1931
(6) Services	11 795	1·557	−0·3625
(7) Imports	16 443	1·000	−1·6799
(8) Depreciation	2 983	0·242	0·0
(9) Net Indirect Taxes	3 094	0·172	0·0
(10) Employees' Income	14 761	1·056	0·0
(11) Profits	11 454	0·774	0·0
(12) Gross National Product Arising [(8) + (9) + (10) + (11)]	32 292	2·243	
(13) Household Consumption	19 537	1·317	
(14) Resid. Final Demand	13 755	0·926	−0·6183
(15) Exports of Ag. etc.	3 532	Nil	
(16) Exports of Metals etc.	2 747	Nil	
(17) Exports of Textiles etc.	790	Nil	
(18) Exports of Mining etc.	2 630	Nil	
(19) Exports of Trade	1 483	Nil	
(20) Exports of Services	3 893	1·000	
(21) Aggreg. of (15) to (20)	15 075	1·000	−0·3625
(22) Total Exports	15 443	1·000	
(23) Import Excess [(7) − (22)]	1 000	0·000	
(24) Expend. on the GNP [(13) + (14) − (23)]	32 292	2·243	

Column (3) of Table 9.10 gives the Shadow Prices of the 13 constraint rows, calculated via the 13 columns of coefficients of the activities selected for the optimal solution. The numerically largest shadow price relates to Imports, Row (7), which shows that a change of one value unit of Imports is worth 1·68 units of Consumption. A unit of Textiles etc. Row (3), has the second largest value, − 0·71, while the Trade row, number (5), has the least non-zero shadow price. We did not allow trade to export in its own right. The second lowest non-zero shadow price is for Services, Row (6), which has a value − 0·36, and this sector has been selected as the optimal exporting sector. The choice can be interpreted as follows. For imports fixed at 1 million guilders, an extra unit of exports of Services will reduce household consumption by only 0·36 units

as against a reduction of 0·71 for an extra unit of Textiles etc. exports, and so on. The zero-level shadow prices for Rows (8) to (11), which take account of the components of GNP Arising, mean that any of these components can increase or decrease by one unit without increasing or decreasing household consumption.

The problem as set up has GNP Arising obtained as the difference between Final Demands and Imports, so that we could omit the 4 Rows (8) to (11) and Columns (7) to (10) from the statement of the problem without in any way affecting the optimal results obtained. These rows and columns are therefore not essential to the statement of the L.P. problem, and to obtain a zero shadow price for such rows is not unreasonable, implying neutrality towards the outcome.

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EXERCISES

Exercise 9.1

(a) Using the layout of Table 9.1, and the technical coefficients for Columns (1) to (3) appearing in Table 9.2, fill in the transactions table corresponding to the L.P. results given in Column (2) of Table 9.4. The columns for Consumption and Capital Formation given in Table 7.1 need scaling up, to give the L.P. levels shown in Rows (8) and (9) of Table 9.4. The Export column is to have the L.P. results for Rows (1) to (3) and the constant Primary Input entries of Table 9.1.

(b) Verify that the output for each of the three productive sectors and the total inputs equals the levels given by the L.P. results, and that each of the four Primary Input Rows likewise has a row aggregate agreeing with that shown in Column (1) of Table 9.4.

(c) Comparison of Column (2) with Column (1) in Table 9.4 shows that the column (1) level of Consumption is 0·96282 times that of Column (2). Scale down Column (2) so that Column (2) consumption matches that of Column (1), and compare results with those of Column (1).

(d) Again using the scale factor 0·96282, scale down the Import row of the table required for (a) above. Compare results with the Import row of Table 9.1 to see how imports for the latter have been re-allocated for the optimal structure of transactions giving the same level of consumption and capital formation (approximately), for a reduced import surplus. Verify that this surplus is 96·282 per cent of £131 million.

Exercise 9.2

(a) Using the coefficients of Column (9), Consumption, given in Table 9.6, and the shadow prices of the Constraint rows given in Table 9.7, Column (3), show that the value of one unit of Column (9) in terms of the shadow price is 0·0872.

(b) Verify that the aggregate value of the coefficients in any one column of the first three columns of Table 9.6, when priced by the shadow prices, is zero, subject to rounding errors.

APPENDIX TO CHAPTER 9
THE MAIN CHARACTERISTICS OF LINEAR PROGRAMMING

The technique of Linear Programming (L.P.) was developed in the late 1940's and is of such power and wide applicability that it requires, and merits, a special course of study on its own. It is also a rather complicated procedure and therefore cannot be covered in a single chapter. All that can be done here is to describe the main features of the technique. Chapter 9 illustrates it by some numerical exercises. Students who master these should be able to solve others of a similar nature. Problems on a small scale, i.e. having only a few variables, can be solved on desk calculators but normal practical problems must be solved on a computer, and the main computer systems all have packages which solve such problems.

It is not necessary to understand computer science in order to solve L.P. problems. If one knows how to present the data to the computer, its package will do the rest. It is, however, essential to understand the L.P. technique sufficiently well to have an intelligent grasp of the meaning of the optimal solution and the stages by which is has been reached.

The General Form of the L.P. Problem

The scheme of objective function and constraints can be set out as follows:

$$\left.\begin{array}{ll} \text{Maximize } \sum_{j=1}^{n} k_j x_j & \text{The Objective Function} \\ \text{subject to } \sum_{j=1}^{n} a_{ij} x_j \leqslant C_i & \text{The Constraints } i = 1, 2, \ldots\ldots p \end{array}\right\} \quad (9.23)$$

where $x_1, x_2 \ldots x_n$ are variables such as sector outputs or activity levels,
$k_1, k_2 \ldots k_n$ are the coefficients of the x's in the objective function,
a_{ij} is the coefficient of x_j in the ith constraint, there being p constraints in the set,
C_i is the constant, possibly zero, for constraint i.

The number of variables, n, exceeds the number of constraints, p. The inequality sign $\leqslant$ (less than or equal to) may in some or all constraints be used as an equality (=); it does not have to be an inequality.

The objective function being a linear function can increase or decrease indefinitely according to the values of its variables. It therefore appears reasonable that a finite algebraic maximum value of the objective function would require finite upper limits for the variables, these conditions being ensured by

the C_i on the right-hand side of each inequality statement being a finite constant.

Expression of a Minimization Problem in the General Form

$$\left.\begin{array}{ll} \text{Minimize} & \sum_{j=1}^{n} k_j x_j \\ \text{subject to} & \sum_{j=1}^{n} a_{ij} x_j \geqslant C_i \\ & i = 1, 2, \ldots p \end{array}\right\} \quad (9.24)$$

can be re-written

$$\left.\begin{array}{ll} \text{Maximize} & (-\sum_{j=1}^{n} k_j x_j) \\ \text{subject to} & (-\sum_{j=1}^{n} a_{ij} x_j) \leqslant (-C_i) \end{array}\right\} \quad (9.25)$$

Thus by multiplying the objective function and inequalities of the form $\geqslant$ (greater than or equal to) by minus one, a minimization problem can be changed into one of maximization. A meaningful solution will be obtained to a meaningful problem and the computer programmes generally have no objection to negative signs before coefficients or constants, although small-problem solutions on desk-machines require positive starting-values which may not be possible if one is using the algebraic system (9.25).

The Solution of the Linear Programming Problem

The first step towards solution is to modify each constraint which has been stated as an inequality, so as to make it an equality. Thus we replace constraint i of (9.23)

$$\sum_{j=1}^{n} a_{ij} x_j \leqslant C_i$$

by

$$\sum_{j=1}^{n} a_{ij} X_j + S_i = C_i \quad (9.26)$$

where S_i is denoted 'slack variable i' and exists only for such constraints as were genuine inequalities. So that generally the number of S_i variables is less than p, the number of constraints. The designation 'slack' refers to the role of this extra variable in absorbing the slackness of the inequality so as to provide an exact equation instead.

Each S_i has zero as its coefficient in the objective function, thus leaving the latter completely unaffected. The system is now ready for solution and has the form:

$$\left.\begin{array}{ll} \text{Maximize} & \sum_{j=1}^{n} k_j x_j + \sum_i 0{\cdot}0\, S_i \\ \text{subject to} & \sum_{j=1}^{n} a_{ij} X_j + 1{\cdot}0\, S_i = C_i \\ & i = 1, 2, \ldots p \end{array}\right\} \quad (9.27)$$

and S_i existing only for some values of i, there being s slacks S_i, in all.

The Special Solution, Non-Optimal, for $(n + s) = p$

Of limited interest is the case where p, the number of constraints, is equal to the total number of variables, $(n + s)$, including the s existing S_i. In this case there is one unique solution, given by

$$\sum_{j=1}^{n} a_{ij}X_j + S_i = C_i \qquad (9.28)$$

$$i = 1, 2, \ldots, p$$

since there are p equations for p variables. No choice of variables is possible as all are required by (9.28), and the objective function has a single possible value $\Sigma\, k_j x_j$ which includes the x_j solutions given by (9.28), with S_i having no effect upon the objective function. The non-optimal problem thus appears as a special case of the general L.P. problem, for $p = (n + s)$.

It might be well to point out here that the matrix of constraint coefficients is typically of the $(I - A)$ input–output format, thus having an inverse.

The General Solution, for $(n + s) > p$

A few technical terms are required, to discuss the approach to the solution and the way of knowing that the maximum has been reached. It is worth mentioning that two features of the process of solution are necessary, in order to find a meaningful outcome. The first feature is that the set of constraints must be independent and consistent, which means that no one of the p constraints can be obtained as a linear combination of the others, and that no variable is simultaneously required to be greater than and less than a constant of the system. It can be taken that in practical examples involving input–output models, independence and consistency are normal. The second feature is that all variables including the slacks must have positive or zero values to provide an acceptable 'feasible' solution at every stage of the maximization process. Any solution containing negative values, although algebraically correct, is rejected as 'infeasible'. At the initial stages of the process to reach the maximum, such solutions may be used just like feasible solutions, with shadow prices and profitabilities as explained below, and will lead to the first feasible solution.

The method of reaching the maximum is to choose p variables out of the total $(n + s)$ and solve the set of p equations given by the constraints in (9.27) for those p selected variables, after setting all the other $(n + s - p)$ variables equal to zero. The p variables chosen are called the 'basis' set and if their solutions are all positive they form a 'feasible basis'. The solution values of each feasible basis yield a value of the objective function, upon substitution. The 'optimal basis' is the set of p variables having positive solutions which yield the maximum value of the objective function. The selection of variables in question, their solution values, and the resulting maximum, are denoted the 'primal solution'. It can happen that an identical maximum value is produced by more than

one single set of p variables. In this case some of the variables outside one optimal basis are interchangeable with some of the variables in this optimal basis in producing the same value of the maximum, and the solution is 'indeterminate', i.e. the choice of p variable for the optimal basis is not unique. How one recognizes an indeterminate solution will be explained below, via 'profitabilities'. It can be a meaningful answer, so long as it is recognized as indeterminate and not mistaken for a unique primal solution. Indeterminate solutions arise whenever the plane of the objective function, in multi-dimensional space, is parallel to the plane of any one of the constraints. All points on a plane of finite area, rather than the single point at a certain intersection of lines, are solutions – the co-ordinates of each point being a numerical solution. All points on a certain straight line of finite length can be another form of indeterminate solution.

How to Move Towards the Maximum

Upon obtaining the first feasible basis, the p chosen variables provide their columns of coefficients in the set of p constraints, along with their coefficients in the preference function, to calculate p Shadow Prices, one such price for each row of the p rows of constraints. The other $(n + s - p)$ variables outside this feasible basis have zero values for this present stage of the process of reaching the optimum and their coefficients can be ignored.

Assume that the first p columns of the constraints have been re-arranged so as to include the coefficients of the p variables in this basis, denoted $y_1, y_2 \ldots y_p$, some being X_i and some being slacks. Likewise the objective function coefficients have been re-ordered so as to relate to $y_j, j = 1, 2, \ldots . p$. In matrix format

$$(P_1, \quad P_2 \ldots . P_p) \begin{pmatrix} a_{11} & a_{12} & a_{1p} \\ a_{21} & \ldots\ldots\ldots & a_{2p} \\ \vdots & & \vdots \\ a_{p1} & & a_{pp} \end{pmatrix} = (k_1, k_2 \ldots . k_p) \qquad (9.29)$$

gives as its solution the vector of shadow prices

$$(P_1, \ldots . P_p) = (k_1, k_2 \ldots . k_p)A^{-1} \qquad (9.30)$$

Since (9.29) can be written

$$\sum_{i=1}^{p} P_i a_{ij} = k_j \qquad j = 1, 2, \ldots p \qquad (9.31)$$

the shadow prices (P_i) value the coefficients in Column j in such a way that the cost of Column j given by the left-hand side of (9.31) is just equal to k_j, the contribution of one unit of y_j to the objective function. The equation set (9.29)

or (9.31), with one side multiplied by minus one, are referred to as the dual and their solution for the optimal basis, given by (9.30) but with sign changed, is called the 'dual solution' by contrast with the primal solution which gives the values of $y_1, y_2 \ldots . y_p$, the set of p variables forming the optimal basis. It is clear from (9.29) that the shadow prices are in the value units used for the objective function and that each selection of p variables has its own set of shadow prices.

Having the shadow prices for one basis of p columns enables the 'Profitability' of all $(n + s)$ columns to be computed. The profitability of Column j is given by

$$k_j - \sum_{i=1}^{p} P_i a_{ij} \tag{9.32}$$

and is referred to in some computer printouts as the 'Reduced Cost'. It is zero for all p columns of the basis used to calculate the shadow prices, via (9.31). In fact (9.31) can be interpreted as the condition of zero profitability, for all p variables in the basis in question being used to compute shadow prices. For the remaining $(n + s - p)$ columns, a negative profitability means that the related variable would reduce the value of the objective function by being introduced into the next basis. A positive profitability, on the contrary, means an increase of the objective function by putting in the variable of the column of coefficients in question in place of one of the variables of the set of p in the present basis. Which variable is to be cast out to make room for the more profitable newcomer* will not be analysed here, but the process of selection is now apparent. The maximum has been reached when all columns outside those in the last basis show negative or zero profitabilities. For all results negative, a unique maximum has been accomplished, with one single set of p variables being the primal solution. For one or more columns outside this basis showing zero profitability, the related variables are equally valid as part of the primal solution and are interchangeable with some of the variables in this primal, to give an identical maximum. We are confronted with an indeterminate solution.

The rationale of the profitability approach is as follows. A shadow price for each row has been calculated by consistently making the costs of inputs, per unit of output, just equal to the contribution by that unit to the objective function, for the set of p activities in the basis being considered. At these prices the coefficients in any column j outside that basis can be priced and summed to give v_j, the aggregate cost per unit of each such activity, at present prices. If the coefficient k_j of the objective function, for that column j, is algebraically greater

* Suppose that Column k, outside the present basis columns 1 to p, shows the maximum profitability, and its coefficients form the column vector A_k. The column vector B_k is calculated as $B_k = [A_{pp}]^{-1} A_k$ where $[A_{pp}]^{-1}$ is the inverse of the matrix of coefficients forming the present basis, A_{pp}. Next, the p ratios (y_i/B_{ki}) are formed, y_i being the ith variable solution given by the present basis and B_{ki} being the ith element of B_k. Let the minimum numerical value of these p ratios occur for the rth ratio. Then variable y_r of the present basis goes out, being replaced by the variable relating to column k, for the next basis.

than the aggregate cost thus calculated, i.e. if $(k_j - v_j)$ is positive, then it is profitable to include Column j, meaning activity j, in the next basis, thus replacing some activity in the most recent basis. Each unit of activity j will increase the objective function above its most recent level.

Two Further Comments

Two further comments complete the discussion of the methodology and ideas. Suppose that Columns 1 to p of the constraint coefficients have been re-ordered so as to relate to the set of p variables forming an optimal basis, denoted $y_1, y_2 \ldots . y_p$, with corresponding re-arrangement of k_j, the coefficients of the objective function. All other coefficients may be ignored, having zero values for their variables. The *primal solution* is given by

$$\begin{bmatrix} a_{11} & a_{12} \ldots a_{1p} \\ \cdot & \cdot \cdot \cdot \cdot \\ \cdot & \cdot \cdot \cdot \cdot \\ \cdot & \cdot \cdot \cdot \cdot \\ a_{p1} & a_{pp} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ y_p \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ \cdot \\ C_p \end{bmatrix} \qquad (9.33)$$

to yield values of $y_1, y_2 \ldots . y_p$.

The *dual solution*, inherent already in (9.29), uses the identical set of a_{ij} coefficients but is minus one times the set of shadow prices. Thus

$$(-d_1, -d_2 \ldots -d_p) \begin{bmatrix} a_{11} & a_{12} \ldots a_{1p} \\ \cdot & \cdot \cdot \cdot \cdot \\ \cdot & \cdot \cdot \cdot \cdot \\ \cdot & \cdot \cdot \cdot \cdot \\ a_{p1} \cdot & \cdot \ldots a_{pp} \end{bmatrix} = (k_1, k_2 \ldots . k_p) \qquad (9.29)'$$

gives the corresponding set of negative shadow prices, the dual solution $(d_1, d_2 \ldots . d_p)$. So that the values of the 'dual activity' as printed out by computer programmes need to be multiplied by minus one before use as shadow prices. A like relationship holds between values y_j and shadow prices p_j for each feasible basis, not merely the optimal.

The second comment concerns zero-level shadow prices. If the primal solution contains slack variables S_i, S_j etc., then the corresponding constraint rows i, j, etc. have zero shadow prices. The other constraint rows do not have zero shadow prices.

Equation (9.31) above shows that the shadow prices satisfy for column j the equality

$$\sum_{i=1}^{p} P_i a_{ij} = k_j \qquad (9.31)'$$

The column for each slack variable has k_j zero since, as stated above, the slacks have no effect upon the objective function, having been included merely to fill out the inequalities. The only coefficient in constraint rows 1 to p of the column

of a slack variable is 1·0 in Row i. Thus (9.31) reduces to

$$1{\cdot}0 P_i = 0{\cdot}0 \qquad (9.34)$$

which gives $P_i = 0$ for each slack appearing in the primal. That is, Row i in these circumstances has a shadow price of zero. Although the main interest lies in a zero value of P_i for the primal solution, each slack variable appearing in any feasible basis on the way to the maximum is associated with zero shadow price of its row.

It might be worth mentioning that for shadow price calculations, variables or activities which are not slacks may behave like slacks and yield zero shadow prices. Any variable which has 0·0 as its coefficient of the objective function and only a single coefficient non-zero in its column in the set of constraints will yield a zero value of its shadow price. This result is apparent from the reasoning of the previous paragraph. It can be interpreted as meaning that the variable in question is passive or neutral in the determination of the optimum – an identical result would be obtained by omission of its row and column from the system. Its output level could be obtained, after solution of the reduced system, by applying its row coefficients to the solution values.

MATHEMATICAL APPENDIX: MATRIX ALGEBRA

Introduction

Matrix algebra originated from the necessity of solving simultaneous linear equations and of dealing in a compact way with various algebraic transformations. It would be very tedious whenever we had occasion to manipulate sets of equations or to refer to properties of the coefficients, if we had to write either the equations or the coefficients in full. For this reason an algebraic system has been developed whereby whole sets of numbers can be represented by simple symbols.

Suppose in ordinary algebra there are two relations:

$$3x_1 + 4x_2 = y_1$$

$$5x_1 + 7x_2 = y_2$$

By detaching the coefficients from the variables on the left-hand side, this system of equations can be written in the following way:

$$\begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

In matrix notation the array of coefficients which is called a matrix can be written as A, the column of x's can be written as X, and the column of y's as Y, and the whole expression as:

$$AX = Y$$

We can immediately realise the enormous economy of writing when there are many relations and many variables. Not alone is there economy of writing, however, there are other advantages as well, for the array of coefficients represented by A can be regarded as an operator acting on the variables x_1 and x_2 in much the same way as in ordinary algebra 'a' acts on 'x' to produce 'ax'. Rules have been developed for such operations and it is this body of such rules that makes up the algebra of matrices. It should be stated, however, that the operation of matrices does not reduce the volume of actual calculations on any problem, but it does lend order to the calculations.

Though the rules of ordinary and matrix algebra are similar in many respects all the rules for the operation of expressions representing single numbers do not hold good for expressions representing sets of numbers. Hence a separate algebra

of matrices has to be developed and the student must become familiar both with the notation used and with the rules involved.

Definition of a Matrix

A matrix is a set of figures or symbols (positive, negative, or zero) arranged in rows and columns in rectangular form. The array of numbers represented by the letter B below is a matrix having two rows and three columns:

$$B = \begin{bmatrix} 5 & -2 & 0 \\ 4 & 3 & 6 \end{bmatrix}$$

The individual numbers are called the elements of the matrix. Thus figure 4 at the junction of the second row and first column is an element of B. In matrix algebra any ordinary number such as 1, 2, 3 etc. is called a *scalar*, hence the elements of B are all scalars.

In making general statements about a matrix it is usual to replace the numerical elements by letters with double subscripts. Thus A is a matrix with the same number of rows and columns as B but with letters rather than numbers for the different elements.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

The subscripts indicate the location of the different elements in the matrix. The first subscript always indicates the row and the second subscript indicates the column. Thus a_{13} is the element in the first row and third column of matrix A. Similarly, a_{21} is the element in the second row and the first column and so on for the other symbols. In general any element of a matrix is indicated by the symbol a_{ij} where the subscript i refers to the row and the subscript j refers to the column.

The *order* of a matrix is an indication of the number of rows and columns it contains. Since A above has two rows and three columns it is a (2×3) matrix. In general, the number of rows in a matrix is indicated by the letter 'm' and the number of columns by the letter 'n' and the order by $(m \times n)$. A scaler is a (1×1) matrix, i.e. a matrix having one row and one column.

Matrix Notation

Normally in matrix operation a single capital letter such as A is used to represent a matrix so that instead of having to write out the whole array of numbers each time we simply use one capital letter to represent them. Sometimes, however, in order to distinguish between different matrices a particular matrix may be represented by a capital letter with subscripts, i.e.

$$A = [A_{ij}] \; (i = 1, 2 \ldots . m; j = 1, 2 \ldots . n)$$

where A_{ij} refers to a matrix having i rows ranging from 1 to m and j columns ranging from 1 to n. In cases, however, where no confusion can arise it is better to omit the brackets and suffices altogether and to represent the matrix by a single capital letter.

Vectors

A matrix having a number of elements arranged in a single column, i.e. an $(m \times 1)$ matrix, is referred to as a column matrix, or more usually as a column vector. As a column of numbers takes up a good deal of vertical space, column vectors are usually written as a single row of elements enclosed in face brackets. Thus:

$$\{d_{11} \quad d_{21} \quad d_{31}\} = \begin{bmatrix} d_{11} \\ d_{21} \\ d_{31} \end{bmatrix} = D_i \, (i = 1 \ldots 3)$$

A column vector such as this can be written in abbreviated form as $\{d_i\}$ $(i = 1, 2 \ldots m)$ showing that the vector has 'i' rows ranging from 1 to m.

In the same way a matrix with only a single row of elements is called a row vector. When it is necessary to write a row vector at length the usual square brackets are used but special brackets $\lfloor \; \rfloor$ are used to denote a row vector in abbreviated form, e.g.

$$\lfloor dij \rfloor = [d_{11} \quad d_{12} \quad d_{13}] = D_j \, (j = 1, \ldots 3)$$

The above symbols for vectors can often be quite confusing and where possible it is best to adopt the following system of symbols.

(*a*) Capital letters without subscripts to represent both matrices and vectors. The particular type of matrix or vector will be clear from the context.

(*b*) Small letters with subscripts to represent individual elements of matrices or vectors. Thus where A is a matrix, a_{ij} is the element in the ith row and the jth column. If A is a column vector, a_i is the typical element, and if A is a row vector a_j is the typical element.

Transposition of Matrices

When the rows and columns of a matrix are interchanged we obtain the transpose of the matrix. The transpose of matrix A is written as A'. The rows of A are the columns of A' while the columns of A are the rows of A'. Thus if:

$$A = \begin{bmatrix} 3 & -2 & 4 \\ 0 & 3 & 5 \end{bmatrix}; \qquad A' = \begin{bmatrix} 3 & 0 \\ -2 & 3 \\ 4 & 5 \end{bmatrix}$$

It should be noted that whereas A is a (2×3) matrix, A' is a (3×2) matrix, and in general we can say that if A is an $(m \times n)$ matrix its transpose A' is an $(n \times m)$ matrix. Thus

$$A = \begin{bmatrix} a_{11} & a_{12} \ldots\ldots & a_{1n} \\ a_{21} & a_{22} \ldots\ldots & a_{2n} \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} \ldots\ldots & a_{mn} \end{bmatrix} \text{ and } A' = \begin{bmatrix} a_{11} & a_{21} \ldots\ldots & a_{m1} \\ a_{12} & a_{22} \ldots\ldots & a_{m2} \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ a_{1n} & a_{2n} & a_{mn} \end{bmatrix}$$

In a similar manner the transpose of a column vector is a row vector and the transpose of a row vector is a column vector. Thus if:

$$D = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 4 \end{bmatrix}; \; D' = [2 \quad 3 \quad 0 \quad 4]$$

Two operations of transposition restore the original matrix. Thus $(A')' = A$. The operation is therefore said to be reflexive.

Special Matrices

Square Matrices

When the number of rows and columns in a matrix are equal, i.e. when $m = n$, the matrix is said to be square. C below is a square matrix having 3 rows and 3 columns.

$$C = \begin{bmatrix} 4 & 1 & 7 \\ 3 & 2 & 5 \\ -2 & 6 & 8 \end{bmatrix}$$

For square matrices, which are of special importance in input–output analysis, the elements denoted x_{ii} define the principal diagonal, i.e. the diagonal going from top left to bottom right. Thus if:

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$$

the elements forming the principal diagonal are x_{11}, x_{22} and x_{33}.

Symmetric Matrices

A symmetric matrix is a square matrix in which the column elements above the principal diagonal are equal to the corresponding row elements below the diagonal. E below is a symmetric matrix.

$$E = \begin{bmatrix} 3 & 2 & 4 & 0 \\ 2 & 6 & -1 & 7 \\ 4 & -1 & 5 & 0 \\ 0 & 7 & 0 & 1 \end{bmatrix}$$

As can be seen, the principal diagonal reads 3, 6, 5, 1. The element in the second column of the first row is 2 and is the same as that in the first column of the second row, which is also 2. Similarly, the element in the third column of the first row, which is 4, is equal to that in the first column of the third row, which is also 4. In general, it can be seen that for this matrix $e_{ij} = e_{ji}$ and because of this equality, a symmetric matrix is identical with its transpose. $E \equiv E'$.

If $E' = -E$ so that $e_{ji} = -e_{ij}$ the matrix E is said to be skew symmetric. Such a matrix must have zero elements in the principal diagonal, since $e_{ii} = -e_{ii}$. E following is a skew symmetric matrix.

$$E = \begin{bmatrix} 0 & 3 & 2 \\ -3 & 0 & 5 \\ -2 & -5 & 0 \end{bmatrix}$$

Diagonal Matrices

A square matrix in which all the elements except those in the principal diagonal are zero is called a diagonal matrix. Matrices A and B below are diagonal matrices. In A the elements in the principal diagonal are not equal while in B these elements are equal

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}; \qquad B = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

A diagonal matrix, such as B, in which all the elements in the principal diagonal are equal is called a *scalar matrix.*

Unit Matrix

A special form of scalar matrix is the unit matrix, also called the identity

matrix, which is a square matrix in which all the elements in the principal diagonal are unity and all other elements are zero. The unit matrix, which is of particular importance in mathematical analysis, is usually indicated by the letter I.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Null Matrix

If all the elements of a matrix are zero it is called a null matrix, which is always indicated by 0, thus:

$$0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Though this may appear to be a trivial type of matrix it has many useful applications in mathematical analyses.

Operation of Matrices

It has been found that with certain exceptions the matrix symbols behave as if they were ordinary algebraic symbols. Thus concepts of addition, subtraction, multiplication and (in a certain limited degree) division of matrices are meaningful. Before going on to discuss these concepts however, it is necessary to state three basic laws governing the operation of ordinary algebraic symbols. Though these laws are so ingrained that they appear self evident, nevertheless, they do not always hold true in matrix algebra. Following are the laws:

(1) *Commutative law:* $a + b = b + a$ and
$ab = ba$

(2) *Associative law:* $a + (b + c) = (a + b) + c$ and
$a(bc) = (ab)c$

(3) *Distributive law:* $a(b + c) = ab + ac$

Having stated these laws, we are now in a position to see if, and when, they hold true in matrix operations.

Equality of Matrices

Two matrices are equal if they are of the same order and if the individual elements are equal, element by element. Thus if:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

then $a_{11} = b_{11}$; $a_{12} = b_{12}$; $a_{21} = b_{21}$ and $a_{22} = b_{22}$.
In general, therefore, if $A = B$, $a_{ij} = b_{ij}$.

Addition and Subtraction

Addition and subtraction of matrices can be performed only if the matrices are of the same order, i.e. have the same number of rows and columns. Addition is performed by adding the corresponding elements of the matrices. Thus:

$$\begin{bmatrix} 4 & -2 & 1 \\ 0 & 3 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 5 & -4 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 4 \\ 5 & -1 & 7 \end{bmatrix} \qquad (a.1)$$

Subtraction is simply the reverse of addition and follows the same rules. In matrix notation, if the sum of two matrices A and B is defined as the matrix C whose typical element

$$c_{ij} = a_{ij} + b_{ij}$$

then

$$C = A + B$$

Similarly, if the difference of two matrices A and B is defined as the matrix D, whose typical element

$$d_{ij} = a_{ij} - b_{ij},$$

then

$$D = A - B$$

Since any element in a matrix sum is equal to the algebraic sum of the corresponding elements in the summed matrices, the addition of matrices is subject to the same laws as the addition of scalars, by assuming of course that the elements are scalars. Hence, the commutative and associative laws of addition and subtraction of scalars hold good.

In other words, if we add two matrices A and B to obtain C the result is the same if we obtain the elements of C as $c_{ij} = a_{ij} + b_{ij}$ or as $b_{ij} + a_{ij}$. Similarly, if we add three matrices A, B and C to obtain D the result is the same if we obtain the elements of D as

$$d_{ij} = a_{ij} + (b_{ij} + c_{ij}) \text{ or as } (a_{ij} + b_{ij}) + c_{ij}$$

Typical Input–Output Matrices

A typical matrix in input–output analysis is known as an $(I - A)$ matrix. This matrix is obtained by subtracting a matrix of technical coefficients A from I which is a unit matrix of the same order. If the matrix of technical coefficients is

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(I - A) = \begin{bmatrix} (1-a_{11}) & -a_{12} & -a_{13} \\ -a_{21} & (1-a_{22}) & -a_{23} \\ -a_{31} & -a_{32} & (1-a_{33}) \end{bmatrix} \qquad (a.2)$$

Hence once the elements of A are known, $(I - A)$ can be obtained by deducting the diagonal elements of A from 1 and multiplying all the other elements by -1.

Multiplication of a Matrix by a Scalar

In matrix algebra the simplest form of multiplication is multiplication of a matrix by a scalar or ordinary number. Multiplication of a matrix by a scalar, k, written either before or after the matrix, is equivalent to multiplication of every element in the matrix by k. Thus:

$$k \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{bmatrix}$$

If k is taken as 3 and numbers substituted for the a_{ij}'s in the above matrix we obtain

$$3 \begin{bmatrix} 2 & 0 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 9 & -12 \end{bmatrix}$$

In the multiplication of a matrix by a scalar or scalars, the three basic laws for multiplication of ordinary numbers hold true

(*a*) *Commutative law:* $kA = Ak$
(*b*) *Associative law:* $(kc)A = k(cA)$
(*c*) *Distributive law:* (1) $k(A + B) = kA + kB$ and
(2) $kA + cA = A(k + c) = (k + c)A$

where small letters represent scalers and capital letters represent matrices.

The first of these laws is self-evident and readers can easily verify that the other two hold in individual cases. Thus:

$$(b) \quad (2 \times 3) \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix} = 6 \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 6 & -12 \\ 18 & 30 \end{bmatrix} = 2 \begin{bmatrix} 3 & -6 \\ 9 & 15 \end{bmatrix}$$

$$(c.1) \quad 2 \left[\begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ -4 & 3 \end{bmatrix} \right] = 2 \begin{bmatrix} 3 & -1 \\ -1 & 8 \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ -2 & 16 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 6 & 10 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ -8 & 6 \end{bmatrix}$$

$$(c.2)\ 2\begin{bmatrix}1 & -2\\3 & 5\end{bmatrix} + 3\begin{bmatrix}1 & -2\\3 & 5\end{bmatrix} = \begin{bmatrix}2 & -4\\6 & 10\end{bmatrix} + \begin{bmatrix}3 & -6\\9 & 15\end{bmatrix} = \begin{bmatrix}5 & -10\\15 & 25\end{bmatrix}$$

$$= 5\begin{bmatrix}1 & -2\\3 & 5\end{bmatrix} = \begin{bmatrix}1 & -2\\3 & 5\end{bmatrix}5$$

Multiplication of Matrices

Two matrices can only be multiplied if the number of rows of one is equal to the number of columns of the other. When this condition is satisfied the matrices are said to be *conformable*. Specifically, if A is an $(m \times n)$ matrix and B is an $(n \times p)$ matrix, the product AB is an $(m \times p)$ matrix. This rule can be expressed simply as follows:

$$(m \times n) \times (n \times p) \rightarrow (m \times p)$$

Hence if A has 2 rows and 3 columns and B has 3 rows and 4 columns the product AB will have 2 rows and 4 columns, i.e.

$$(2 \times 3) \times (3 \times 4) \rightarrow (2 \times 4)$$

It should be noted: (i) the middle term is common to the two matrices being multiplied, and if this is not so the multiplication cannot be performed, (ii) the order of the matrix obtained by multiplication is given by the two outside terms. Since vectors are matrices with either single rows or columns the rules for matrix multiplication apply also to multiplication of vectors by vectors or to vectors by matrices. The results of such multiplications are discussed later.

In matrix multiplication the rows of the left-hand matrix are multiplied by the columns of the right-hand matrix and not the other way round.

Example

$$\overset{A}{\begin{bmatrix}3 & 2 & 1\\4 & 0 & 2\end{bmatrix}} \overset{B}{\begin{bmatrix}2 & 3\\0 & 2\\5 & 6\end{bmatrix}} = \overset{C}{\begin{bmatrix}11 & 19\\18 & 24\end{bmatrix}}$$

The result is arrived at as follows:

1st row of A x 1st col. of B = $(3 \times 2) + (2 \times 0) + (1 \times 5) = 11$
1st row of A x 2nd col. of B = $(3 \times 3) + (2 \times 2) + (1 \times 6) = 19$

2nd row of A x 1st col. of B = $(4 \times 2) + (0 \times 0) + (2 \times 5) = 18$
2nd row of A x 2nd col. of B = $(4 \times 3) + (0 \times 2) + (2 \times 6) = 24$

It should be noted that in this example we multiplied a (2×3) matrix by a (3×2) matrix and obtained a (2×2) matrix, i.e.

$$(2 \times 3) \times (3 \times 2) = (2 \times 2)$$

In symbolic form, if

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \text{ then}$$

$$AB = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

where

$$\begin{aligned} c_{11} &= a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} \\ c_{12} &= a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ c_{21} &= a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} \\ c_{22} &= a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{aligned}$$

In general terms we can say that if two matrices A and B are multiplied together to give $AB = C$ the element in the i th row and j th column of the product matrix C is obtained by multiplying the elements in the ith row of A into the corresponding elements in the jth column of B and summing the products so obtained. In symbols

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

where:
c_{ij} is any element in the product matrix C, and a_{ik} and b_{kj} are elements in the original matrices A and B. The k subscript indicates that the column number from which the a_{ij} is taken is the same as the row number from which the b_{ij} is taken. The notation below and above the summation sign Σ indicates that the summation of products ranges from $k = 1$ to $k = n$, n being the number of columns of A and rows of B.

Matrix multiplication can be simplified somewhat by transposing matrices, for example, in multiplying A and B above. Matrix A can be transposed to enable a column-by-column multiplication. Similarly, B can be transposed to enable a row-by-row multiplication.

Premultiplication and Postmultiplication

Multiplication of ordinary numbers follows the commutative law, i.e. the product ab is the same as the product ba. Matrix products on the other hand

are not generally commutative. The product AB is not equal to BA except in special cases. The product AB is referred to as the *premultiplication* of B by A or as the *postmultiplication* of A by B.

The difference between AB and BA may arise for a number of reasons.
(1) One of the products may not be defined at all because while two matrices A and B may be conformable for premultiplication of A by B they may not be conformable for postmultiplication of A by B. For example if A is an $(m \times n)$ matrix and B is an $(n \times p)$ matrix B can be premultiplied by A giving an $(m \times p)$ matrix, i.e. AB multiplication requires

$$(m \times n) \times (n \times p) \rightarrow mp$$

In this case however, if $m \neq p$, A cannot be premultiplied by B. The product $(n \times p) \times (m \times n)$ does not exist because the middle term is not common.
(2) Though two matrices A and B may be conformable for premultiplication and postmultiplication, i.e. reverse multiplication, AB may not be equal to BA because they are of different orders. Two matrices are conformable for reverse multiplication if the number of rows in the one is equal to the number of columns in the other and if the number of columns in the one is equal to the number of rows in the other. Thus if A is an $(m \times n)$ matrix and B is an $(n \times m)$ matrix

$$AB \rightarrow (m \times n) \times (n \times m) \rightarrow m \times m$$
$$BA \rightarrow (n \times m) \times (m \times n) \rightarrow n \times n$$

and as can be seen AB is an $(m \times m)$ matrix and BA is an $(n \times n)$ matrix. These matrices are therefore not of the same order (except in the special case where $m = n$) and hence cannot be equal.
(3) Square matrices of the same order are conformable for reverse multiplication, but even in such cases AB is not generally the same as BA. For example, suppose A and B which are two square matrices of the same order have the elements shown below

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & 4 \\ 3 & 1 \end{bmatrix}$$

Reverse multiplication of these matrices gives the following results:

$$AB = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 5 & 4 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 21 & 14 \\ 17 & 8 \end{bmatrix}$$

$$BA = \begin{bmatrix} 5 & 4 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 19 & 26 \\ 10 & 10 \end{bmatrix}$$

showing that in this case $AB \neq BA$.

Commutative Matrices

If $AB = BA$, a condition which only holds in special cases, the matrices A and B are said to be commutative. Matrices can be commutative only if they are conformable and square. Examples of commutative matrices are:
(a) A square matrix A and a conformable unit matrix I,
(b) Two conformable scalar matrices B and C,
(c) Any scalar matrix D and other conformable matrix E,
(d) Two conformable diagonal matrices,
(e) Any matrix A and its inverse A^{-1} (see later for definition of A^{-1}).
Examples of (a), (b), (c), and (d) above are given below.

(a) Multiplication of a Square Matrix by a Unit Matrix

$$AI = IA = \begin{bmatrix} 3 & 0 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 4 & 2 \end{bmatrix}$$

It should be noted that this result is true only if I and A are square matrices of the same order. It is not true that $AI = IA$ for A of order $m \times n$. In that case I would need to be of order $n \times n$ on its first appearance and of order $m \times m$ on its second. It should be stated however that a matrix of any order is left unchanged when it is either premultiplied or postmultiplied by a conformable unit matrix. Hence in matrix algebra multiplicationf of a matrix by I is equivalent to multiplication by unity in ordinary algebra.

(b) Multiplication of Scalar Matrices

$$BC = CB = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

It should be noted that the product $BC = CB$ is a scalar matrix whose diagonal elements are the products of the diagonal elements of A and B i.e. $3 \times 2 = 6$.

(c) Multiplication of Any Matrix by a Scalar Matrix

$$DE = ED = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -4 \\ -12 & 16 \end{bmatrix}$$

(d) Multiplication of Two Diagonal Matrices

$$PQ = QP = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 15 & 0 \\ 0 & 12 \end{bmatrix}$$

Products of Several Matrices

The product of several matrices can be illustrated by reference to three matrices A, B and C. To get the product of ABC in this order it is necessary for A to be conformable with B and for AB to be conformable with C. ABC exists only if the orders of the different matrices are:

$$A = (m \times n): \quad B = (n \times p): \quad C = (p \times q)$$

Here, as in the case of two conformable matrices, the middle terms are common, and, as in that case also, the final product is given by the extreme terms. Hence the order of ABC is $(m \times q)$ as shown in the following examples where A is a (2×2) matrix, B is a (2×3) matrix, C is a (3×2) matrix and ABC is a (2×2) matrix.

$$A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} : \quad B = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \end{bmatrix} : \quad C = \begin{bmatrix} 2 & 4 \\ 0 & -3 \\ 1 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 6 & 1 \\ -1 & -2 & 1 \end{bmatrix}$$

$$(AB)C = \begin{bmatrix} 9 & 6 & 1 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & -3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 19 & 20 \\ -1 & 4 \end{bmatrix} \qquad (a.3)$$

Associative Law for Multiplication

The associative law for multiplication of ordinary numbers applies also to the multiplication of several matrices. For ordinary numbers $abc = a(bc) = (ab)c$. The same holds for matrices. The product of three matrices A, B and C is:

$$ABC = (AB)C = A(BC)$$

This result can be verified using the matrices in $(a.3)$ above as follows:

$$BC = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & -3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 8 & 16 \end{bmatrix}$$

$$ABC = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 8 & 16 \end{bmatrix} = \begin{bmatrix} 19 & 20 \\ -1 & 4 \end{bmatrix} \qquad (a.4)$$

As can be seen ABC is the same in $(a.3)$ and $(a.4)$. The product D of three

matrices A, B, C, of order $(m \times n)$, $(n \times p)$ and $(p \times q)$ respectively can be expressed in symbols as follows:

$$d_{ij} = \sum_{h=1}^{n} a_{ih}\left(\sum_{k=1}^{p} b_{hk}c_{kj}\right)$$

where small letters with subscripts are elements of the corresponding matrices represented by capital letters. The k subscript indicates that the column number from which b_{ij} is taken is the same as the row number from which c_{ij} is taken. Similarly, the h subscript indicates that the column number from which a_{ij} is taken is the same as the row number of the element in BC by which a_{ij} is multiplied.

The double summation over np terms indicated here can equally well be carried out in different orders to yield exactly the same result and normally it is usual to indicate the double summation indiscriminately by

$$\sum_{h=1}^{n} \sum_{k=1}^{n} a_{ih}\, b_{hk}\, c_{kj}$$

Distributive Law for Addition and Multiplication of Matrices

It was shown above that the distributive law holds true for addition of matrices and multiplication by a scalar, i.e. $k(A + B) = kA + kB$, where k is a scalar and A and B are matrices of the same order. The distributive law also holds true for addition and multiplication of matrices provided of course that the matrices are conformable for this purpose. Thus:

$$A(B + C) = AB + AC$$

Readers can verify by using a practical example that the result holds true.

Other Rules

(a) Cancellation of matrices is not generally possible. For example if $AB = 0$ ($0 =$ a null matrix of appropriate order) it does not follow that either $A = 0$ or $B = 0$. A numerical example explains

$$\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

(b) If A is conformable with B and with C so that $AB = AC = 0$ it does not follow that $B = C$, e.g.

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} : \; B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} : \; C = \begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix} :$$

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

$$AC = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Hence though $AB = AC = 0, B \neq C$.* This difficulty does not arise, however, in the reverse case. The product of any matrix and a conformable zero matrix is always zero.

Multiplication of Vectors and Matrices

The same rules apply to vectors as to matrices but some of the results are so useful that they require special treatment.

(a) Product of Two Vectors

Two column vectors are not conformable for multiplication; neither are two row vectors. Column and row vectors are conformable if the number of rows in the column vector is equal to the number of columns in the row vector.

Multiplication of two conformable vectors gives a product which is either a scalar or a matrix. Premultiplication of a column vector by a conformable row vector gives a scalar product, e.g.

$$[1 \quad 3 \quad 4] \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} = 11$$

Premultiplication of a row vector by a conformable column vector gives a matrix:

$$\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} [1 \quad 3] = \begin{bmatrix} 3 & 9 \\ 0 & 0 \\ 2 & 6 \end{bmatrix}$$

(b) Product of a Matrix and a Vector

A column vector can be premultiplied but cannot be postmultiplied by a matrix. Similarly, a row vector can be postmultiplied but cannot be premultiplied by a matrix.

Premultiplication of a column vector by a matrix if they are conformable gives a column vector having the same number of rows as the matrix. The matrix A and vector D are conformable if A has order $(m \times n)$ and D has order

* It should be stated that in cases such as this there is a relationship between B and C, B being some scalar multiple of C.

$(n \times 1)$, i.e. $(m \times n) \times (n \times 1) \rightarrow (m \times 1)$. Thus:

$$AD = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 12 \end{bmatrix}$$

Premultiplication of a matrix by a row vector if they are conformable gives a row vector having the same number of columns as the original matrix. The matrix B and the vector F are conformable if B has order $(p \times q)$ and F has order $(1 \times p)$, i.e. $(1 \times p) \times (p \times q) \rightarrow (1 \times q)$. Thus:

$$FB = [1 \quad 3 \quad 4] \begin{bmatrix} 2 & 0 & -1 \\ 3 & 2 & 0 \\ 1 & 4 & 5 \end{bmatrix} = [15 \quad 22 \quad 19]$$

The Transpose of a Product

The transpose of matrix products is not the product of the transposed matrices, i.e. $(AB)' \neq A'B'$. In fact if B is premultiplied by A to give C, the transpose of C is equal to the transpose of A premultiplied by the transpose of B, i.e. $(AB)' = B'A'$. Similarly, $(ABC)' = C'B'A'$. This result, which can easily be verified, is known as the *reversal rule for the products of transposed matrices.*

Matrix Inversion

Generally speaking one matrix cannot be divided by another. There are, however, matrix operations which play roles corresponding to division in ordinary algebra if the divisor is a square non-singular matrix (the term non-singular is explained later). The operation of "division" in matrix algebra has its parallel in ordinary algebra. Suppose we have an equation $3x = 6$ then $x = 6/3$ or $(1/3 \times 6)$. In the latter case 6 is multiplied by 1/3 which is the reciprocal or inverse of 3. Since, however, $1/3 = 3^{-1}$ we can also write our solution as: $x = 3^{-1} \times 6$.

The same procedure is used for division in matrix algebra. If we wish to divide one matrix by another we multiply the one by the reciprocal or inverse of the other. Thus, if we wish to divide a matrix B by another matrix A, the result C is written as

$$C = A^{-1} B$$

It should, however, be kept in mind that C cannot be defined unless A^{-1} exists and is conformable with B. As stated above A^{-1} can only exist if A is a *square non-singular matrix.*

Multiplication of a Matrix by its Inverse

We know that in ordinary algebra $a^{-1} \times a = a^0 = 1$. Similarly, in matrix algebra, any matrix A and its inverse A^{-1}, when multiplied together, give the matrix equivalent of unity, i.e.

$$A^{-1}A = I$$

where I is the unit matrix. Furthermore, A and A^{-1} obey the commutative laws of multiplication so that $A^{-1}A = AA^{-1} = I$. Hence both premultiplication and postmultiplication of any matrix by its inverse gives the unit matrix.

The procedure of matrix inversion is used mainly in the solution of simultaneous equations and is best explained by reference to such solutions. Suppose we have the following system of simultaneous equations

$$\begin{aligned} 2x_1 + 3x_2 &= 14 \\ 1x_1 + 5x_2 &= 21 \end{aligned} \qquad (a.5)$$

By separating the coefficients from the variables on the left hand side, this system can be written as:

$$\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 14 \\ 21 \end{bmatrix} \qquad (a.6)$$

or in a matrix form as

$$AX = Y \qquad (a.7)$$

where A is the matrix of coefficients; X is the vector representing x_1 and x_2, and Y is a vector representing the numbers 14 and 21. If (a.7) were an ordinary equation the value of X would be determined by dividing across by A giving

$$X = \frac{Y}{A}$$

Since (a.7) is a matrix equation the solution is obtained by multiplying across by A^{-1} so that we obtain

$$A^{-1}AX = A^{-1}Y \qquad (a.8)$$

Now since $A^{-1}A = I$ and since $IX = X$, (a.8) can be reduced to

$$X = A^{-1}Y \qquad (a.9)$$

Hence the numerical values for the elements of X are obtained by premultiplying the vector Y, the elements of which are known by A^{-1}, the inverse of matrix A. To do this the elements of A^{-1}, which are unknown, must be computed from the elements of A, which are known. Various methods are available for calculating the inverse matrix but if the number of variables involved

is more than three the calculations are onerous and are best done on a computer. We show below how the inverse matrix may be computed for the simple (2 x 2) matrix given in (a.6) above. The method used is known as the *equality of matrices* method.

Calculating the Inverse of a (2 x 2) Matrix by the Equality-of-Matrices Method

Let the matrix to be inverted be that given in (a.6) above, i.e.

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix} \qquad (a.10)$$

Let the elements of A^{-1}, which are to be determined, be

$$A^{-1} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \qquad (a.11)$$

now since $A^{-1}A = I$ (the unit matrix)

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (a.12)$$

Multiplying together the two matrices on the left-hand side of (a.12) using the methods of matrix multiplication, we obtain

$$\begin{bmatrix} 2c_{11} + 1c_{12} & 3c_{11} + 5c_{12} \\ 2c_{21} + 1c_{22} & 3c_{21} + 5c_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (a.13)$$

Since the matrices on the left- and right-hand side of (a.13) are equal, element by element, we can obtain two sets of simultaneous equations from the arrays, as follows:

$$\text{(a)} \quad \begin{aligned} 2c_{11} + 1c_{12} &= 1 \\ 3c_{11} + 5c_{12} &= 0 \end{aligned} \qquad \text{(b)} \quad \begin{aligned} 2c_{21} + 1c_{22} &= 0 \\ 3c_{21} + 5c_{22} &= 1 \end{aligned} \qquad (a.14)$$

Solving these equations by ordinary algebraic procedures we obtain $c_{11} = \frac{5}{7}$; $c_{12} = -\frac{3}{7}$; $c_{21} = -\frac{1}{7}$ and $c_{22} = \frac{2}{7}$. The inverse matrix A^{-1} is therefore

$$A^{-1} = \begin{bmatrix} \frac{5}{7} & -\frac{3}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix} \qquad (a.15)$$

Solving the System of Equations

After calculating the numerical elements of the inverse matrix the next exercise is to use these elements in the solution of our system of equations. To do this the elements of X and Y from (a.6) above, and the figures for A^{-1} from (a.15), are substituted into equation (a.9) to give the following result:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{5}{7} & -\frac{3}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix} \begin{bmatrix} 14 \\ 21 \end{bmatrix} \tag{a.16}$$

Multiplying out the figures on the right-hand side of (a.16) we obtain

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \tag{a.17}$$

Indicating that $x_1 = 1$ and $x_2 = 4$. We can verify that this result is correct by substituting these values for x_1 and x_2 into equation system (a.5).

Having worked out an example, we are now in a position to state the general rules for the solution of any system of simultaneous equations which has a solution.

Let the equation in matrix form be

$$AX = Y$$

where X is a vector of unknowns,
A is the matrix of coefficients of the unknowns, and
Y is the vector of given numerical values.

Proceed as follows:

(1) Write out fully the numerical values for the elements of A and Y, and symbolic values for the elements of X, as in (a.6).

(2) Invert matrix A to obtain A^{-1}, and

(3) Premultiply vector Y by A^{-1} to obtain the numerical values for the elements of X (the unknown).

Some Useful Properties of Inverse Matrices

The following properties of inverse matrices, which are given without proof, are so useful in practical work that they should be committed to memory:

(a) The product of a matrix and its inverse is commutative, i.e.

$$A^{-1}A = AA^{-1} = I, \text{ the Unit Matrix.}$$

(b) *Inverse of an Inverse.* The inverse of an inverse matrix is equal to the original matrix, i.e.

$$(A^{-1})^{-1} = A$$

(c) The inverse of I is I itself, i.e. $I^{-1} = I$

(d) *Inverse of matrix product – the reversal rule.* The inverse of a matrix product is the product of the inverses of the separate matrices in inverse order, i.e.

$$(AB)^{-1} = B^{-1}A^{-1}$$

(e) *Inverse of a transpose.* The inverse of a transposed matrix is the transpose of the inverse, i.e.

$$(A')^{-1} = (A^{-1})'$$

(f) The inverse of a diagonal matrix is obtained by getting the reciprocals of the individual elements. Thus

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}^{-1} = \begin{bmatrix} 1/a_{11} & 0 & 0 \\ 0 & 1/a_{22} & 0 \\ 0 & 0 & 1/a_{33} \end{bmatrix}$$

Algebraic Manipulation of Inverse Matrices

Care must be taken, in manipulating inverse matrices, to ensure that multiplication by an inverse is carried out in the correct order on both sides of the sign of equality, e.g. if

$$AX = Y$$

then

$$A^{-1}AX = A^{-1}Y$$

but

$$A^{-1}AX \neq YA^{-1}$$

unless Y and A^{-1} are commutative with respect to multiplication. Indeed if Y is a column vector then YA^{-1} does not exist. However, if $BA = C$ where A, B and C are matrices, then

$$BAA^{-1} = CA^{-1}$$

but

$$BAA^{-1} \neq A^{-1}C$$

unless A^{-1} and C are commutative.

A more difficult case is the following. If $ABC = P$, where A, B, C, and P are matrices, and we wish to find AC, we cannot do so directly by bringing the inverse of B across to the right-hand side of the sign of equality. We must proceed as follows:

$$\begin{aligned} ABCC^{-1} &= PC^{-1} \\ AB &= PC^{-1} \\ ABB^{-1} &= PC^{-1}B^{-1} \\ A &= PC^{-1}B^{-1} \\ AC &= PC^{-1}B^{-1}C \end{aligned}$$

If, however, $ABC = P$, where B and C are diagonal matrices and we want to find AC, we can do so directly, since, as shown above, two conformable diagonal matrices are commutative for multiplication. Hence

$$ABC = ACB \qquad ACBB^{-1} = PB^{-1}$$

i.e.

$$ACB = P \qquad AC = PB^{-1}$$

Note on Matrix Inversion

As stated above, numerous methods are available for the inversion of matrices, all of which are very tedious if the matrix to be inverted is of the order (4 x 4) or larger. It is not proposed to explain methods of inversion of large matrices here as any worthwhile treatment of the subject would require far more space than is available. Nowadays, practical research workers have their matrices inverted on a computer and are not very concerned with the arithmetical details of the operation. However, for the readers who wish to pursue the matter further, various methods of inversion are explained in the references at the end of this appendix. Heady and Candler [6] give a detailed explanation of a method known as the Crout method, which is convenient for the inversion of relatively small matrices on a desk calculator. Dwyer [2], in his book *Linear Computations*, gives numerous methods suitable for large models, as do also Frazer, Duncan and Collar [5]. Some of the matrices encountered in Input–Output Analysis are often too large for inversion on the available computers and in such cases the inversions have to be done by what Dwyer [2] calls "extension methods", i.e. by manipulation and inversion of submatrices. As these methods are rather complicated they are not explained here, but we explain in the next section a method, suitable for the inversion of small models, based on determinants.

Determinants

It was stated above that the matrix operation corresponding to division could not be performed unless the divisor was a *non-singular*, square matrix. In other words, the inverse of a matrix cannot be calculated unless the matrix has these properties. The definition of a square matrix has already been explained but not that of a non-singular matrix. In order to understand the meaning of this latter term it is necessary to be familiar with *determinants.*

Relationship between Matrices and Determinants

As explained previously, a matrix is an array of elements written in rows and columns. It has no single numerical value. Certain square matrices, however, possess associated numerical values, called determinants.

The idea of a determinant is best understood by reference to the solution of simultaneous equations. Let us consider the solution of such equations in one and two unknowns.

The solution of $a_1x_1 = y$ is $x_1 = y_1/a_1$ provided that $a_1 \neq 0$. The solution of

$$a_{11}x_1 + a_{12}x_2 = y_1 \qquad (a.18)$$

$$a_{21}x_1 + a_{22}x_2 = y_2$$

found by eliminating first x_1 and then x_2 is

$$x_1 = \frac{y_1a_{22} - y_2a_{12}}{a_{11}a_{22} - a_{12}a_{21}}$$
$$x_2 = \frac{y_2a_{11} - y_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}} \qquad (a.19)$$

which can be written in the form

$$\frac{x_1}{y_1a_{22} - y_2a_{12}} = \frac{x_2}{y_2a_{11} - y_1a_{21}} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \qquad (a.20)$$

The equation system (a.18) can be written in matrix form as $AX = Y$, i.e.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad (a.21)$$

If we consider the term $(a_{11}a_{22} - a_{12}a_{21})$ of the solution (a.20), this is obtained directly from the matrix A in (a.21), by cross multiplication of the diagonal terms

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

and is known as the determinant of A. The determinant of A is usually written as $\det A$ or $|A|$.

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

A matrix and its determinant must be carefully distinguished. A matrix is an array of elements, while the determinant of a matrix is a scalar value associated with that matrix. Vertical bars are used to denote a determinant. The denominator of x_1 in (a.20), i.e. $(y_1a_{22} - y_2a_{12})$, is obtained from the matrix A by writing the vector of y's in the first column of the A matrix and considering the determinant of the resulting matrix, i.e.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \rightarrow \begin{bmatrix} y_1 & a_{12} \\ y_2 & a_{22} \end{bmatrix} \rightarrow \begin{vmatrix} y_1 & a_{12} \\ y_2 & a_{22} \end{vmatrix} = y_1a_{22} - y_2a_{12}$$

Similarly, the denominator of x_2 in (a.20), i.e. $(y_2a_{11} - y_1a_{21})$ is obtained by writing the vector of y's for the second column of the A matrix, i.e.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \rightarrow \begin{bmatrix} a_{11} & y_1 \\ a_{21} & y_2 \end{bmatrix} \rightarrow \begin{vmatrix} a_{11} & y_1 \\ a_{21} & y_2 \end{vmatrix} = a_{11}y_2 - a_{21}y_1$$

The solution (a.20) can be written as

$$\frac{X_1}{\begin{vmatrix} y_1 & a_{12} \\ y_2 & a_{22} \end{vmatrix}} = \frac{X_2}{\begin{vmatrix} a_{11} & y_1 \\ a_{21} & y_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \qquad (a.22)$$

This method of solution is known as Cramer's rule for the solution of a system of simultaneous equations and can be extended to larger systems of equations. At this point, it should be stated that a determinant can only be defined for a square matrix but, as will be shown later, not all square matrices have non-vanishing determinants, i.e. determinants whose values are not zero. For example, the determinant of the matrix in (a.10) above is:

$$\det \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix} = (2 \times 5) - (3 \times 1) = 7 \qquad (a.23)$$

whereas

$$\det \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} = (2 \times 6) - (3 \times 4) = 0$$

It should by now be obvious to readers that there is a close connection between the determinant and the inverse of a matrix, since both may be used in the solution of the same system of equations. Further evidence of this relationship is provided by the value of the determinant of the matrix in (a.23) above (i.e. 7) which is the denominator of the elements of the inverse of this matrix in (a.15). This relationship is, however, not entirely straightforward. The inverse of a matrix cannot be computed by just dividing the elements of the matrix by its determinant. Certain operations have to be performed on the original matrix before dividing by the determinant in order to obtain the inverse matrix. These operations will be explained later.

It is a simple matter to obtain the determinant of a (2 x 2) matrix, i.e. a second-order determinant. For larger matrices the computations are more involved, but as will be shown, the general determinant is simply the extension of the cross product rule from a (2 x 2) to an (n x n) matrix.

The determinant of a (3 x 3) matrix may be calculated as in (a.24) below

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$
$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \qquad (a.24)$$

As can be seen, the third-order determinant may be calculated by multiplying each element in the first row of the matrix by a second-order determinant and summing the results with appropriate alternating positive and negative signs. The second-order determinant multiplying a_{11} is obtained from the original matrix by omitting the first row and column (i.e. the row and column in which a_{11} is located). The second-order determinant multiplying a_{12} is obtained by omitting the first row and second column of the original matrix (i.e. the row and column in which a_{12} is located), and similarly for the second-order determinant multiplying a_{13}.

As is apparent from the last expression in (a.24) above, the third-order determinant is a way of writing an algebraic sum of six terms, each term being the product of three elements of the matrix, so chosen that one element comes from each row and one from each column. Also, half the terms have positive signs and half have negative signs.

Some Characteristics of Determinants

(a) If two rows or two columns of a matrix are equal or proportional to one another, the value of the determinant of this matrix is zero. For example, the second row of the matrix in (a.23) is twice that of the first row and as can be seen the value of the determinant is zero.

$$\det \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} = (2 \times 6) - (3 \times 4) = 0 \qquad (a.25)$$

(b) If a determinant is multiplied by a constant, then one row (or column) is multiplied by this constant.

$$\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 5$$

$$3\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = \begin{vmatrix} 6 & 3 \\ 3 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 9 & 12 \end{vmatrix} = \begin{vmatrix} 6 & 1 \\ 9 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 3 & 12 \end{vmatrix} = 15$$

This is in contrast with the property of a matrix, since if a matrix is multiplied by a constant then every element of the matrix is multiplied by this constant. This can cause some confusion initially. Consider

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \text{ a matrix}$$

$$3A = \begin{bmatrix} 6 & 3 \\ 9 & 12 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 5$$

$$|3A| = \begin{vmatrix} 6 & 3 \\ 9 & 12 \end{vmatrix} = 45$$

while

$$3|A| = 15$$

Hence

$$3|A| \neq |3A|$$

(c) The value of the determinant of a matrix is unchanged when a multiple of the elements of one row (or of one column) is added to the corresponding elements of a second row (or a second column) of the matrix. Thus

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix} = -4 \qquad (a.26)$$

If we now add the first row to the second row of (a.26) we obtain the following matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 4 \\ 1 & 1 & 2 \end{bmatrix}$$ which also has a determinant of -4.

(d) The interchange of two rows or two columns of a matrix may change the sign of the determinant of the matrix, depending on the number of interchanges involved. If an even number of interchanges are carried out, the sign of the determinant is unchanged, while if an odd number of interchanges are made, the sign is changed. Taking the determinant in (a.26) above, whose value is -4, and interchanging the first two rows, the reader can verify that the value of the new determinant is $+4$. If however the first and third rows are interchanged the value remains as -4; similarly for interchange of columns. Thus:

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{vmatrix} = -1 \begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

(e) The determinant is unchanged if the rows and columns of a matrix are transposed, i.e.

$$\det \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} = \begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix} = -10$$

(f) If all the elements in any row or column of a matrix are zero, the determinant is zero. This result is self-evident since the determinant has in each term one element of each row and one element of each column as factors of that term. Hence, if all the elements in any row or column are zero the determinant must be zero.

Minors and Co-factors

As explained above, the determinant of a matrix can be evaluated by multiplying the elements of any row or column by lower-order determinants obtained by omitting in turn the rows and columns in which the elements are located. A lower-order determinant can be defined for every element in a matrix by deleting the row and column intersecting in the element. The derived determinant is called the minor of the selected element and if the matrix is of order n the minor is of order $(n - 1)$. The minor of any element a_{rs} is denoted by Δ_{rs}.

The co-factor of a selected element is the minor of the element with a sign attached. The rule of signs is as follows: if the numbers of the subscripts of a selected element add to an even number then a plus sign is given to the co-factor, and if the subscripts add to an odd number, a minus sign is allotted. Thus the sign of the co-factor of a_{rs} is positive if $(r + s)$ is even and it is negative if $(r + s)$ is odd. There are as many co-factors as there are elements in a matrix and they can be arranged in matrix order. Thus if the matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

the matrix of co-factors is

$$A_{rs} = \begin{bmatrix} \Delta_{11} & -\Delta_{12} & \Delta_{13} \\ -\Delta_{21} & \Delta_{22} & -\Delta_{23} \\ \Delta_{31} & -\Delta_{32} & \Delta_{33} \end{bmatrix}$$

The transpose of the matrix of co-factors is called the *adjugate* or *adjoint matrix* of A, i.e.

$$[A_{rs}]' = \begin{bmatrix} \Delta_{11} & -\Delta_{21} & \Delta_{31} \\ -\Delta_{12} & \Delta_{22} & -\Delta_{32} \\ \Delta_{13} & -\Delta_{23} & \Delta_{33} \end{bmatrix}$$

and if each element in this adjoint matrix is divided by the determinant of A, given that $|A| \neq 0$, a third matrix is formed which is the inverse of A, i.e. A^{-1}. Thus

$$A^{-1} = \begin{bmatrix} \dfrac{A_{rs}}{|A|} \end{bmatrix}' = \frac{1}{|A|} \begin{bmatrix} \Delta_{11} & -\Delta_{21} & \Delta_{31} \\ -\Delta_{12} & \Delta_{22} & -\Delta_{32} \\ \Delta_{13} & -\Delta_{23} & \Delta_{33} \end{bmatrix}$$

if $|A| \neq 0$.

Calculation of the adjoint matrix and division of the elements of this by the determinant is therefore another method of evaluating the inverse matrix. It might be mentioned in passing that this is a very useful method of inverting small matrices, and in fact forms the basis for many of the methods used on large matrices.

We are now in a position to refer back to the matrix in (a.14), in order to show how it can be inverted, using the adjoint method. The matrix is:

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$$

The matrix of co-factors, obtained by dropping the rows and columns in which each element is located, is:

$$A_{rs} = \begin{bmatrix} 5 & -1 \\ -3 & 2 \end{bmatrix}$$

The adjoint matrix is

$$[A_{rs}]' = \begin{bmatrix} 5 & -3 \\ -1 & 2 \end{bmatrix}$$

The determinant of A has already been shown to be 7, hence the inverse matrix is

$$A^{-1} = \begin{bmatrix} \dfrac{A_{rs}}{|A|} \end{bmatrix}' = \frac{1}{7} \begin{bmatrix} 5 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \dfrac{5}{7} & -\dfrac{3}{7} \\ -\dfrac{1}{7} & \dfrac{2}{7} \end{bmatrix} \qquad (a.27)$$

This is the same inverse matrix as that obtained previously by the equality-of-matrices method and given in (a.15) above.

Inversion of a (3 x 3) input–output matrix by use of the adjoint matrix

We show below how the (3 x 3) $(I-A)$ matrix given in (2.12) of the text was inverted, using the adjoint method. The $(I-A)$ matrix is:

$$(I-A) = \begin{bmatrix} 0{\cdot}9891 & -0{\cdot}1518 & -0{\cdot}0038 \\ -0{\cdot}1383 & 0{\cdot}8178 & -0{\cdot}0845 \\ -0{\cdot}0550 & -0{\cdot}0599 & 0{\cdot}9353 \end{bmatrix} \qquad (a.28)$$

The minors of the different elements are:

$$\Delta_{11} = (\ 0{\cdot}8178 \times \ 0{\cdot}9353) - (-0{\cdot}0845 \times -0{\cdot}0599) = 0{\cdot}75983$$
$$\Delta_{12} = (-0{\cdot}1383 \times \ 0{\cdot}9353) - (-0{\cdot}0845 \times -0{\cdot}0550) = -0{\cdot}13400$$
$$\Delta_{13} = (-0{\cdot}1383 \times -0{\cdot}0599) - (\ 0{\cdot}8178 \times -0{\cdot}0550) = 0{\cdot}05326$$

$$\Delta_{21} = (-0{\cdot}1518 \times \ 0{\cdot}9353) - (-0{\cdot}0038 \times -0{\cdot}0599) = -0{\cdot}14221$$
$$\Delta_{22} = (\ 0{\cdot}9891 \times \ 0{\cdot}9353) - (-0{\cdot}0038 \times -0{\cdot}0550) = 0{\cdot}92490$$
$$\Delta_{23} = (\ 0{\cdot}9891 \times -0{\cdot}0599) - (-0{\cdot}1518 \times -0{\cdot}0550) = -0{\cdot}06760$$

$$\Delta_{31} = (-0{\cdot}1518 \times -0{\cdot}0845) - (-0{\cdot}0038 \times \ 0{\cdot}8178) = 0{\cdot}01593$$
$$\Delta_{32} = (\ 0{\cdot}9891 \times -0{\cdot}0845) - (-0{\cdot}0038 \times -0{\cdot}1383) = -0{\cdot}08411$$
$$\Delta_{33} = (\ 0{\cdot}9891 \times \ 0{\cdot}8178) - (-0{\cdot}1518 \times -0{\cdot}1383) = 0{\cdot}78789$$

The determinant of $(I-A)$ is

$$|(I-A)| = (0{\cdot}9891 \times 0{\cdot}75983) - (-0{\cdot}1518 \times -0{\cdot}13400) + (-0{\cdot}0038 \times 0{\cdot}05326) = 0{\cdot}731005$$

The matrix of co-factors, i.e. minors with appropriate signs, is

$$(I-A)_{rs} = \begin{bmatrix} 0{\cdot}75983 & 0{\cdot}13400 & 0{\cdot}05326 \\ 0{\cdot}14221 & 0{\cdot}92490 & 0{\cdot}06760 \\ 0{\cdot}01593 & 0{\cdot}08411 & 0{\cdot}78789 \end{bmatrix} \qquad (a.29)$$

The adjoint matrix is

$$[(I-A)_{rs}]' = \begin{bmatrix} 0{\cdot}75983 & 0{\cdot}14221 & 0{\cdot}01593 \\ 0{\cdot}13400 & 0{\cdot}92490 & 0{\cdot}08411 \\ 0{\cdot}05326 & 0{\cdot}06760 & 0{\cdot}78789 \end{bmatrix} \qquad (a.30)$$

The inverse of $(I-A)$ is obtained by dividing every element in (a.30) by the determinant which is 0·731005. Hence

$$(I-A)^{-1} = \left[\frac{(I-A)_{rs}}{|(I-A)|}\right]' = \begin{bmatrix} 1\cdot03943 & 0\cdot19454 & 0\cdot02179 \\ 0\cdot18331 & 1\cdot26524 & 0\cdot11506 \\ 0\cdot07286 & 0\cdot09247 & 1\cdot07782 \end{bmatrix} \quad (a.31)$$

As a check on the work this result should be multiplied by the basic $(I-A)$ matrix from (a.28). The arithmetic is correct if the multiplication gives a unit matrix, beeause

$$(I-A)(I-A)^{-1} = I$$

Thus

$$\begin{bmatrix} 0\cdot9891 & -0\cdot1518 & -0\cdot0038 \\ -0\cdot1383 & 0\cdot8178 & -0\cdot0845 \\ -0\cdot0550 & -0\cdot0599 & 0\cdot9353 \end{bmatrix} \begin{bmatrix} 1\cdot03943 & 0\cdot19454 & 0\cdot02179 \\ 0\cdot18331 & 1\cdot26524 & 0\cdot11506 \\ 0\cdot07286 & 0\cdot09247 & 1\cdot07782 \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot0000 & 0\cdot0000 & 0\cdot0000 \\ 0\cdot0000 & 1\cdot0000 & 0\cdot0000 \\ 0\cdot0000 & 0\cdot0000 & 1\cdot0000 \end{bmatrix} \quad (a.32)$$

Singular Matrices and Linear Dependence

A square matrix A is singular if its determinant vanishes, i.e. $|A| = 0$. The inverse of such a matrix cannot therefore be defined, i.e. A^{-1} does not exist. Conversely, a square matrix A is non-singular if its determinant is not equal to zero, i.e. $|A| \neq 0$. If a matrix is singular, its vectors (either rows or columns) are said to be linearly dependent, and conversely if the vectors of a matrix are linearly dependent, the matrix is singular.

A system of vectors is linearly dependent if one of them can be expressed as some linear combination of the others. If none of the vectors of a matrix can be expressed as a combination of the others the vectors are said to be linearly independent, and if it is a square matrix its determinant is not zero and so its inverse can be defined.

A more precise definition of linear dependence is that there can be found scalars $k_1, k_2 \ldots k_n$, not all zero, such that their products, with the vectors of a given matrix, add to zero. The column vectors of matrix R below are obviously linearly dependent because the elements of the third column are twice those of the second.

$$R = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 4 \\ 0 & 3 & 6 \end{bmatrix}$$

so that

$$0\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 1\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which can be written as

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 4 \\ 0 & 3 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Also it can be shown that $|R| = 0$ since

$$0\begin{vmatrix} 2 & 4 \\ 3 & 6 \end{vmatrix} - 1\begin{vmatrix} 1 & 4 \\ 0 & 6 \end{vmatrix} + 2\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = 0$$

Therefore R^{-1} does not exist.

Linear dependence may exist, however, even though it is not obvious. For example, the determinant of matrix B below is zero, though proportionality between either the rows or columns is not evident.

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

If this matrix is examined closely, it will be seen that the difference between the second and first row is the same as that between the third and second row, i.e. Row 2 − Row 1 = Row 3 − Row 2 = (3 3 3) hence 2(Row 2) − Row 1 − Row 3 = 0, the actual equality being

$$2[4 \quad 5 \quad 6] - 1[1 \quad 2 \quad 3] - 1[7 \quad 8 \quad 9] = [0 \quad 0 \quad 0]$$

Hence the rows of B are linearly dependent and accordingly the matrix is singular.

In general terms if $(v_1), (v_2) \ldots (v_n)$ are the column vectors of an $(n \times n)$ matrix A, and if scalars $k_1, k_2 \ldots k_n$, not all zero, can be found, such that $k_1(v_1) + k_2(v_2) + \ldots + k_n(v_n) = 0$, then $(v_1), (v_2) \ldots (v_n)$ are linearly dependent.

7. Carry out the following multiplications

(*a*) $$\begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 6 \end{bmatrix}$$

(*b*) $$\begin{bmatrix} -4 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -3 & -2 \\ 1 & 4 \end{bmatrix}$$

(*c*) $$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 5 \\ -1 & 2 \end{bmatrix}$$

(*d*) $$\begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 3 & 4 \\ 2 & 9 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$$

8. If $A = \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix}$ and $B \begin{bmatrix} -1 & 2 \\ 3 & 5 \end{bmatrix}$

find AB and BA.
What can you say about the product of A and B?

9. Write down two matrices whose product is commutative.

10. Write down the value of

(*a*) $(3 \quad 2 \quad 1)\begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$ and (*b*) $\begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}(3 \quad 2 \quad 1)$

11. (*a*) Write down the value of

$$\begin{bmatrix} 3 & 2 & 1 \\ 4 & 5 & 2 \\ 3 & 2 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

(*b*) Write the following system of equations in matrix form:

$$\begin{aligned} 3x + 2y + z &= 11 \\ 4x + 5y + 2z &= 19 \\ 3x + 2y + 5z &= 23 \end{aligned}$$

What is the solution of this system?
Using (*a*) and (*b*) write down, by observation, a solution of the system of equations in (*b*).

12. If $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & 2 \\ 5 & 7 \end{bmatrix}$

find the values of

(a) $(AB)'$

(b) $(ABC)'$

(c) $(BC)'$

13. If $A = \begin{bmatrix} 4 & 2 & 1 \\ 1 & -2 & 3 \\ -3 & 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 5 & 9 \\ 6 & 2 \end{bmatrix}$,

find the value of AB.

14. Find the inverses of the following matrices:

(a) $\begin{bmatrix} 2 & 1 \\ 4 & 9 \end{bmatrix}$; (b) $\begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix}$; (c) $\begin{bmatrix} -1 & 8 \\ 4 & 2 \end{bmatrix}$

15. If $A = \begin{bmatrix} 1 & 2 \\ 6 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} -9 & 2 \\ 6 & -1 \end{bmatrix}$ find AB and BA.

What is the inverse of A and B?

16. Solve the following equations, using the results of Exercise 14:

(a)
$$\begin{aligned} 2x + y &= -3 \\ 4x + 9y &= 6 \end{aligned}$$

(b)
$$\begin{aligned} 3x - y &= 5 \\ 2x - 4y &= 9 \end{aligned}$$

(c)
$$\begin{aligned} x - 8y &= 4 \\ -4x - 2y &= 3 \end{aligned}$$

17. Find the value of

(a) $\begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix}$; (b) $\begin{vmatrix} 3 & 2 \\ 6 & 9 \end{vmatrix}$;

(c) $\begin{vmatrix} -1 & 2 \\ -4 & 3 \end{vmatrix}$; (d) $\begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix}$

18. Find the value of

(a) $\begin{vmatrix} 1 & 4 & 5 \\ 6 & -9 & 8 \\ -2 & 3 & 2 \end{vmatrix}$; (b) $\begin{vmatrix} 3 & -2 & -1 \\ 4 & 6 & -2 \\ 3 & 1 & 5 \end{vmatrix}$

19. Find, by the adjoint method, the inverse of the following matrix

$$\begin{bmatrix} 3 & -2 & 1 \\ 4 & -2 & 3 \\ 1 & 5 & -1 \end{bmatrix}$$

20. Solve the system of equations

$$\begin{aligned} 3x - 2y + z &= 3 \\ 4x - 2y + 3z &= -1 \\ x + 5y - z &= 2 \end{aligned}$$

21. Invert the matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

22. Write the following system of equations in matrix form and solve for x, y and z.

$$\begin{aligned} x + 2y + z &= 1 \\ -x + 2z &= -1 \\ y + z &= 3 \end{aligned}$$

23. (i) Show that the following vectors are linearly dependent

$$\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}; \quad \begin{bmatrix} 4 \\ 3 \\ -2 \end{bmatrix}; \quad \begin{bmatrix} 0 \\ -5 \\ 8 \end{bmatrix}$$

(ii) What can you say about the matrix

$$\begin{bmatrix} 2 & 4 & 0 \\ -1 & 3 & -5 \\ 3 & -2 & 8 \end{bmatrix}$$

(iii) Can you solve the system of equations

$$\begin{aligned} 2x + 4y &= 0 \\ -x + 3y - 5z &= 0 \\ 3x - 2y + 8z &= 0 \end{aligned}$$

If so, give reasons, and solve.

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